RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

January 2010

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

Algebra Prelim **January 2010**

1. Show that if all subgroups of a group G are normal, then $[[G,G],G] = \{1\}$.
(Hint: By considering the action by conjugation of G on a cyclic subgroup $K \lhd G$, show that $[G,G] \leq C_G(K)$.)

2. Which finite groups have exactly two automorphisms?

3. Let R be a commutative ring with unity. Suppose for each $r \in R$, there exists an integer $n_r > 1$ such that $r^{n_r} = r$. Show that every prime ideal in R is maximal.

4. Determine the Jordan form of the $n \times n$ matrix over a field $\mathbb{F}$ whose entries are all 1's. (The answer depends on whether $\mathrm{char}(\mathbb{F})$ divides n.)

5. Let p be a prime number. Let $\mathbb{F}$ be a field whose characteristic is not p which contains a primitive p-th root of unity. Suppose that $a, b \in \mathbb{F}$ are such that $\mathbb{F}[\sqrt[p]{a}] \neq \mathbb{F}[\sqrt[p]{b}]$. Prove that $\mathbb{F}[\sqrt[p]{a}, \sqrt[p]{b}] = \mathbb{F}[\sqrt[p]{a} + \sqrt[p]{b}]$.

6. Show that $x^5 - 4x + 2$ is not solvable by radicals over $\mathbb{Q}$.