*RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!*

# Algebra

# Ph.D. Preliminary Exam

**January, 2009**

*INSTRUCTIONS*:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate program assistant as to which number you have chosen.

**Algebra Prelim** **January, 2009**

**The six problems have equal points. Please do all of them.**

1. Suppose that $G$ acts transitively on sets $X$ and $Y$, where $1 < |X| < |Y| = p$ and $p$ is prime. Show that $G$ is not simple.

2. Suppose that $G$ is a nonabelian group of order $p^e m$ where $p$ is prime, $p$ does not divide $m$, and $p^e$ does not divide $(m-1)!$. Show that $G$ is not simple.

3. Let $R$ be a commutative ring. Show that an ideal $I$ in $R$ is prime if and only if it satisfies both of the following properties:
(a) if $I = I_1 \cap I_2$ for two ideals $I_1, I_2$ in $R$, then either $I_1 = I$ or $I_2 = I$;
(b) if $a \in R$ and $a^n \in I$ for some positive integer $n$, then $a \in I$.

4. Let $R$ be a commutative ring (with unity), and let $f\colon R^n \to R^m$ be an $R$-module homomorphism. Prove that if $f$ is injective, then $n \leq m$.
[Hint: suppose $n > m$ and let $i\colon R^m \to R^n$ be the inclusion given by

$$(r_1, \ldots, r_m) \mapsto (r_1, \ldots r_m, 0, \ldots, 0).$$

Consider the composition $i \circ f$ together with the associated matrix $A$. Use the characteristic polynomial $p_A(x)$ to show that a non-zero multiple of $(1, \ldots, 1)$ is in the image of $A$, giving a contradiction.]

5. Let $A$ and $B$ be commutative rings (with unity), where $B$ is an $A$-algebra. For a $B$-module $M$, an $A$-derivation of $B$ into $M$ is a map $d\colon B \to M$ such that
(a) $d(b + b') = d(b) + d(b')$ for all $b, b' \in B$,
(b) $d(bb') = d(b)\, b' + b\, d(b')$ for all $b, b' \in B$, and
(c) $d(a \cdot 1_B) = 0$ for all $a \in A$.
Now let $K$ be a field and $L$ be a separable field extension of $K$, viewed as a $K$-algebra. Show that for an $L$-module $M$, any $K$-derivation of $L$ into $M$ is the zero map.
[Hint: for $p(x) \in K[x]$, show that for any $\ell \in L$, $d(p(\ell)) = p'(\ell) d\ell$.]

6. Show that the extension $\mathbb{Q}\big[\sqrt{2+\sqrt{2}}\,\big]/\mathbb{Q}$ is Galois with a cyclic Galois group.