RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Exam

January, 2008

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Algebra Prelim **January, 2008**

1. Let G be a nonabelian finite simple group, and let p be a prime divisor of its order $|G|$. Show that if the number of Sylow p-subgroups of G is n, then $|G|$ divides $n!$.

2. Let G be a finite solvable group. Show that

   (a) G has a nontrivial abelian normal subgroup of prime power order, and

   (b) every maximal proper subgroup of G has prime power index in G.

3. Let R be a UFD such that any ideal generated by two elements of R is principal. Prove that R is a PID.

   (Hint: If $a \in I$ is to generate the ideal I, consider what the factorization of a must look like.)

4. Let A be an $n \times n$ matrix over $\mathbb{C}$ such that $\mathrm{tr}(A^k) = 0$ for all $k > 0$. Show that $A^n = 0$. (The trace $\mathrm{tr}(M)$ of a matrix M is the sum of its diagonal entries.)

5. Find the splitting field of $x^4 + x^3 + 1$ over the 32-element field.

6. True or false? Justify your answer.

   (i) Every field extension of degree 2 is Galois.

   (ii) Every algebraically closed field is infinite.

   (iii) If $\alpha = \sqrt[5]{2+i} + \sqrt[5]{2-i}$, then $\mathrm{Gal}(\mathbb{Q}[\alpha]/\mathbb{Q}) \cong S_5$.