1. There exists an injective group homomorphism $\sigma : S_4 \to A_7$ given by

$$\begin{aligned} \sigma((12)) &= (12)(56), \\ \sigma((23)) &= (23)(56), \\ \sigma((34)) &= (34)(56). \end{aligned}$$

List the elements in one Sylow 2-subgroup of S_4 and hence, or otherwise, write down a Sylow 2-subgroup of A_7. Deduce that A_7 contains precisely 315 Sylow 2-subgroups, each of which is self-normalizing. (Hint: each Sylow 2-subgroup of A_7 contains precisely two elements of cycle type $(4, 2, 1)$.)

2. Classify up to isomorphism all groups of order 8. (Your argument should contain full proofs, although you may use general theorems without proof if you state them clearly.)

3. Let S be the subring of the field of fractions of $\mathbb{R}[x]$ consisting of those fractions whose denominators are relatively prime to $x^2 + 1$, i.e., of the form $p(x)/q(x)$ with $q(x)$ relatively prime to $x^2 + 1$.
   - (i) What are the units of S?
   - (ii) Identify the ideals of S.
   - (iii) Is S a unique factorization domain? Explain.
   - (iv) If $\mathbb{R}$ is replaced by $\mathbb{C}$ and the set of rational functions corresponding to S constructed, would it have a unique maximal ideal? Explain.

4. Let R be a ring with identity, 1, and let $f \in R$ be an idempotent (i.e., $f^2 = f$).
   - (i) Show that $Rf = \{rf : r \in R\}$ is a projective left R-module under the action $r_1 \cdot (rf) = (r_1 r)f$.
   - (ii) Now let $R = M_2(\mathbb{C})$, and let M be the left R-module

$$M = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

with the usual action. Prove that M is projective.

5. Let α be a zero of the polynomial $p(x) = x^3 - x - 1$ over $\mathbb{Z}_3$ in some splitting field.
(i) Express the multiplicative inverse of α as a polynomial of minimum degree in α.
(ii) Express the other zeros of $p(x)$ as polynomials of minimum degree in α.
(iii) What is the minimum polynomial $q(x)$ for α^2?
(iv) Express the other zeros of $q(x)$ as polynomials of minimum degree in α.

6. Let $f(x) = x^3 - 5 \in \mathbb{Q}[x]$.
(i) Find a splitting field for f over $\mathbb{Q}$.
(ii) Find the Galois group for f.
(iii) Find all proper, nontrivial subgroups of this Galois group and the fields to which they correspond according to the fundamental theorem of Galois theory.