RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D.
Preliminary Exam

August, 2018

INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

1. Suppose G is a group of order 385.

   (a) Show that G has exactly one Sylow 11-subgroup and that it is a normal subgroup.

   (b) Show that G has exactly one Sylow 7-subgroup and it is contained in the center of G.

2. Let K be a field and let $K[[x]]$ be the ring of formal power series in x over K.

   (a) Show that $\sum_{i=0}^{\infty} a_i x^i$ is a unit if and only if $a_0 \neq 0$.

   (b) Show that every nonzero proper ideal in $K[[x]]$ is generated by x^k for some $k \geq 1$.

3. Let G denote the Galois group of $f(x) = x^5 - 10x + 5$ over the rationals. View G as a subgroup of S_5.

   (a) Consider any irreducible polynomial $g(x)$ over $\mathbb{Q}$ of prime degree p. Show that the Galois group of $g(x)$ has an element of order p.

   (b) Show that G contains a 5-cycle.

   (c) Show that G contains a 2-cycle (hint: how many real roots does f have?).

   (d) Use the previous results to prove that $G \cong S_5$.

4. (a) Prove that there are exactly two distinct automorphisms of the ring $\mathbb{F}_5 \times \mathbb{F}_{25}$.

   (b) Let $f(x) = x^3 + 3$. Prove that there are exactly two distinct isomorphisms

   $$\rho : \mathbb{F}_5[x]/(f(x)) \to \mathbb{F}_5 \times \mathbb{F}_{25}.$$

   (Hint: Factor $f(x)$ modulo 5.)

5. Let $G = \mathbb{Z}_{p^2}$, the cyclic group of order p^2, where p is an odd prime. Classify all semi-direct products $G \rtimes G$ up to isomorphism.

6. Consider the conjugacy classes in the group $GL(2, \mathbb{F}_5)$ of 2×2 invertible matrices over $\mathbb{F}_5$. How many such conjugacy classes contain matrices whose eigenvalues lie in $\mathbb{F}_5$?