RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

August 2017

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

Algebra Prelim **August 2017**

1. Assume that G is an infinite nonabelian group whose proper subgroups are finite. Show that every proper normal subgroup of G is contained in the center of G. Explain why $G/Z(G)$ is an infinite simple group whose proper subgroups are finite.

2. Suppose that the alternating group A_4 acts transitively on a set X. What are the possible sizes of X?

3. Let A be an integral domain containing a field $\mathbb{F}$ as a subring. This makes A a vector space over $\mathbb{F}$. Show that if A is finite dimensional over F, then A is a field. Show that A need not be a field if it is not finite dimensional over $\mathbb{F}$.

4. You are given that G is a group for which there exist a surjective homomorphism $\alpha\colon \mathbb{Z}^n \to G$ and an injective homomorphism $\beta\colon \mathbb{Z}^n \to G$. What are the possible isomorphism types for G?

5. Consider the following three rings

$$\mathbb{F}_3[x]/(x^2+1), \quad \mathbb{F}_3[x]/(x^2+2), \quad \text{and} \quad \mathbb{F}_3[x]/(x^2+2x+2),$$

where $\mathbb{F}_3$ is the field with 3 elements.

(a) Show that each of these rings is a product of fields, and say which fields are involved.

(b) For each pair of isomorphic rings on this list, provide an explicit isomorphism.

6. Let $p \geq 5$ be a prime number and let L be the splitting field of $x^p - 1$ over $\mathbb{Q}$.

(a) Find explicit generators for $\mathrm{Gal}(L/K)$, and explain why your answer is correct. What is the structure of this group?

(b) Use (a) to find explicit generators for a subfield K of L such that $[L : K] = 2$, and explain why your answer is correct.