*RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!*

# Algebra
# Ph.D. Preliminary Exam

**August 2016**

*INSTRUCTIONS*:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

**Algebra Prelim** **August 2016**

1. Prove that, up to isomorphism, there is a unique group of order 1001 ($= 7 \times 11 \times 13$).
2. Let $S_n$ be the symmetric group on $n$ symbols.
   (i) Prove that if $2 \le n \le 4$ then there is a surjective homomorphism of groups from $S_n$ to $S_{n-1}$.
   (ii) Prove that if $n \ge 5$ then there is no surjective homomorphism of groups from $S_n$ to $S_{n-1}$.
3. Let $R$ be a commutative ring with identity.
   (i) Suppose that $I$ is an ideal of $R$ that is contained in the principal ideal $\langle a \rangle$. Show that there is an ideal $J$ of $R$ such that $I = \langle a \rangle J$.
   (ii) Now suppose that $R = \mathbb{C}[x, y]$. Give an example of two ideals $I \subseteq A$ of $R$ for which there is no ideal $J$ satisfying $I = AJ$.
4. Let $F$ be a field and let $A \in M_n(F)$ be a non-invertible $n \times n$ matrix over $F$.
   (i) Prove that if 0 is the only eigenvalue of $A$ in $F$, and $F$ is algebraically closed, then we have $A^n = 0$.
   (ii) Find an example of a field $F$ and a non-invertible matrix $A \in M_n(F)$ such that 0 is the only eigenvalue of $A$ in $F$, but such that we do not have $A^n = 0$.
5. Let $L/K$ be a Galois extension of fields. The *norm* map from $L$ to $K$ is defined to be

$$N(\alpha) = \prod_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\alpha).$$

   (i) Show that $N$ restricts to a homomorphism of groups from $L^*$ to $K^*$.
   (ii) Let $\mathbb{F}_q$ denote the field with $q$ elements and let $m$ be a positive integer. Show that $N : \mathbb{F}_{q^m}^* \to \mathbb{F}_q^*$ is surjective. [Hint: use the Frobenius automorphism.]
   (iii) Let $\sigma$ be a generator for $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$. Compute the cardinality of

$$S = \left\{ \frac{\alpha}{\sigma(\alpha)} \;\middle|\; \alpha \in \mathbb{F}_{q^m}^* \right\}.$$

   (iv) Show that $\ker(N) = S$, where $N$ and $S$ are as defined in parts (ii) and (iii) respectively.
6. Let $f = x^4 - 3$. Find the degree of the splitting field of $f$ over $\mathbb{Q}$. Describe the Galois group of $f$, by giving its action on the roots of $f$ explicitly, and identifying it as isomorphic to a known finite group.