RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra

Ph.D. Preliminary Exam
Department of Mathematics
University of Colorado Boulder

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INSTRUCTIONS:

1. Answer each of the six questions on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Label each answer sheet with the problem number.

3. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

Q.1 Let G be a group. Let $H \lhd G$ be a normal subgroup of prime index p. Let $a \in H$. Suppose the conjugacy class of a inside G is of size m. Show that the conjugacy class of a inside H is of size either m or m/p.

Q.2 Classify all groups of order 253 up to isomorphism.

Q.3 Let R be a commutative ring with identity and I and J two ideals such that $I + J = R$.

(a) Show that $IJ = I \cap J$.

(b) Give an example where $I + J \neq R$ and $IJ \neq I \cap J$.

Q.4

(a) Suppose that A is a complex $n \times n$ matrix with $A^3 = -A$. Show that A is diagonalizable.

(b) Suppose that that A is a 2×2 matrix over the field $\mathbb{Q}$ of rational numbers with no non-trivial eigenvectors with entries in $\mathbb{Q}$, and that $A^3 = -A$. Show that A is similar over $\mathbb{Q}$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Q.5 Find the number of monic irreducible sextic polynomials in $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the field of three elements.

Q.6 Let $\mathbb{Q}$ be the field of rational numbers, and $\mathbb{C}$ the field of complex numbers. Let $\sqrt{2}$ denote the positive square root of 2 in $\mathbb{C}$. Let $\alpha = \sqrt{4 + 3\sqrt{2}}$ denote the positive square root of $4 + 3\sqrt{2}$ in $\mathbb{C}$.

(a) Determine the minimal polynomial of α.

(b) Show that $L = \mathbb{Q}(\alpha)$ is not galois over $\mathbb{Q}$.

(c) Let M be the galois closure of L over $\mathbb{Q}$. What is the order of the galois group G of M over $\mathbb{Q}$?