Department of Mathematics, University of Colorado Boulder

Algebra Preliminary Examination

August 2013

The six problems have equal points. Please do all of them.

1. Let G be a finite group, and let p be the smallest prime dividing $|G|$, the order of G.
   (a) Prove that every subgroup of G of index p is normal in G.
   (b) Suppose also that G has distinct subgroups H and K, each of which is a simple subgroup of index p. Prove that G is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and determine the number of index p subgroups of G.

2. In this problem m and n are positive integers and p is a prime.

   For any ring A with identity 1_A, let $\mathrm{GL}_m(A)$ denote the group of invertible $m \times m$ matrices with entries in A. If $\varphi : A \to B$ is a homomorphism of commutative rings with $\varphi(1_A) = 1_B$, there is a function $\mathrm{GL}_m(A) \to \mathrm{GL}_m(B)$ by applying φ to the individual entries of a matrix in $\mathrm{GL}_m(A)$. You may use that this is a homomorphism without proof.
   (a) Let $q : \mathrm{GL}_m(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{Z}/p^n\mathbb{Z})$ be the homomorphism associated to the reduction map $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$. Show that q is surjective.
   (b) Show that the kernel of q is isomorphic as a group to the additive group $M_m(\mathbb{Z}/p\mathbb{Z})$ of $m \times m$ matrices with entries in $\mathbb{Z}/p\mathbb{Z}$.
   (c) Compute the order of $\mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})$.
   (d) Compute the order of $\mathrm{GL}_m(\mathbb{Z}/p^n\mathbb{Z})$ for all m and n.

3. Suppose that A is a commutative ring with identity and a *unique* maximal ideal P, and that M is a *finitely generated* A-module such that $PM = M$. Show that $M = \{0\}$, and show by example that the hypothesis that M be finitely generated is necessary.

4. Let M be an $n \times n$ invertible matrix with entries in an algebraically closed field k.
   (a) Show that there exist matrices S and U with all of the following properties: (i) S is diagonalizable; (ii) all the eigenvalues of U are equal to 1; (iii) $M = SU = US$.
   (b) Suppose that M^2 is the identity matrix. Show that if the characteristic of k is not 2, then M is diagonalizable. Give a counterexample to show that this is not necessarily true if k has characteristic 2.

5. Let F and E be fields and let D be an intermediate ring such that $F \subset D \subset E$. Show that if $[E : F]$ is finite, then D is a field. Give a counterexample to show that this is not always true if $[E : F]$ is infinite.

6. Recall that a regular n-gon is constructible by compass and straightedge if and only if the real part of $\zeta = e^{2\pi i/n}$ can be expressed using the field operations and square roots.
   (a) Prove that the regular 17-gon is constructible by compass and straightedge.
   (b) Prove that the regular 19-gon is not constructible by compass and straightedge.