Algebra Exam: Do all six problems.

1. Show that there is no simple group of order 120.

2. Show that any group of order $104 = 2^3 \times 13$ is solvable (without using Burnside's $p^a q^b$ theorem).

3. Suppose R is a commutative ring with identity. A proper ideal I in R is said to be a *primary* ideal if whenever elements a and b in R satisfy $ab \in I$ and $a \notin I$ then there exists a positive integer m such that $b^m \in I$.

   Part a. Show that every prime ideal in R is a primary ideal.

   Part b. Let I be a primary ideal and let

   $$I' = \{a \in R : a^m \in I \text{ for some positive integer } m\}.$$

   Show that I' is a prime ideal containing I.

   Part c. Show that if R is a PID then any primary ideal of R is a power of a prime ideal.

4. Suppose K is a field and that $B \in M_n(K)$ is an $n \times n$ matrix with entries from K. Assume that B has distinct eigenvalues all of which lie in K. Prove that

   $$\{A \in M_n(K) : AB = BA\}$$

   is a vector space over K of dimension n.

5. Let q be a positive integral power of a positive prime integer p and let F_q denote the field with q elements. Prove that the polynomial $T^{q^t} - T$ equals the product of the distinct monic irreducible polynomials $P(T) \in F_q[T]$ with $\deg(P)$ dividing t. (Hint. If you want you may use the fact that the field F_{q^t} consists precisely of the zeros of the polynomial $f(T) = T^{q^t} - T$.)

6. It is a well-known theorem that the Galois group of $X^n - 1$ over $\mathbf{Q}$ is isomorphic to $(\mathbf{Z_n})^*$.

   Part a. Without using the above theorem prove directly that Galois group, G, of $X^8 - 1$ over $\mathbf{Q}$ is isomorphic to $(\mathbf{Z_8})^*$.

   Part b. For each of the subgroups of G from Part a *explicitly* find the corresponding fixed field.