RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

August 2011

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

Algebra Prelim **August 2011**

1. Let $Z(G)$ denote the center of the group G. Show that if $|G : Z(G)| = n > 1$, then any conjugacy class of G has at most $n/2$ elements.

2. Let $T_n(F)$ be the group of invertible $n \times n$ upper triangular matrices over some finite field F of characteristic p. Let $D_n(F)$ be the subgroup of invertible diagonal matrices over F, and let $U_n(F)$ be the *normal* subgroup of $T_n(F)$ consisting of the upper triangular matrices with 1's on the diagonal.

(a) Show that $T_n(F)$ is a semidirect product of its subgroups $U_n(F)$ and $D_n(F)$.

(b) Show that, for every prime q dividing the order of $T_n(F)$, the number of Sylow q-subgroups of $T_n(F)$ is a power of p.

3. Let p and q be distinct prime integers such that neither of the congruences

$$x^2 \equiv p \pmod{q} \qquad \text{and} \qquad x^2 \equiv -p \pmod{q}$$

are solvable in the integers. Show that $\mathbb{Z}[\sqrt{pq}]$ is not a unique factorization domain. (Hint: Make use of the norm, $N(x + \sqrt{pq}\,y) = (x + \sqrt{pq}\,y)(x - \sqrt{pq}\,y) = x^2 - pqy^2$.)

4. Describe all $\mathbb{Z}[i]$-modules of size 100 up to isomorphism. Explain why your list is complete and irredundant. You may use without proof the fact that the cyclic module $\mathbb{Z}[i]/(a + bi)$ has size $a^2 + b^2$.

5. For a Galois extension E/F with Galois group G the norm of an element $u \in E$ is defined by $N_{E/F}(u) = \prod_{\gamma \in G} \gamma(u)$. Hilbert's Theorem 90 states that if E/F is a cyclic extension with Galois group $\langle \sigma \rangle$, then $N_{E/F}(u) = 1$ iff there exists $v \in E$ such that $u = v/\sigma(v)$. Use this to prove that if E/F is an extension of finite fields, then $N_{E/F} \colon E^* \to F^*$ is surjective.

6. Let $f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$.

(a) Show that f is irreducible in $\mathbb{Q}[x]$.

(b) Show that $f(x)$ divides $f(x^2 - 2)$.

(c) Explain why if $\alpha \in \mathbb{C}$ is a root of f, then $\mathbb{Q}[\alpha]/\mathbb{Q}$ is a normal field extension of degree three.