RETURN THIS COVER SHEET WITH YOUR EXAM AND SOLUTIONS!

Algebra
Ph.D. Preliminary Exam

August 2010

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

2. Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (1234 for instance) and notify the graduate secretary as to which number you have chosen.

3. There are 6 problems, each worth the same number of points. Please do them all.

Algebra Prelim **August 2010**

1. In this problem, you may assume without proof that the alternating groups A_n are simple for $n \geq 5$.

   (a) Prove that any index 6 subgroup of the alternating group A_6 is isomorphic to A_5.

   (b) Show that any simple group of order 60 is isomorphic to A_5.

2. Let G be a finite group with a normal subgroup $N \triangleleft G$, and suppose $\theta : G \to H$ is a group homomorphism into a solvable group H. Show that if the commutator subgroup of G/N is itself, then $\theta(G) = \theta(N)$.

3. Show that x, y and z are irreducible and prime elements of $k[x, y, z]$, where k is a field. Prove that $k[x, y, z]/\langle xy - z^2\rangle$ is an integral domain.

4. Let p be an odd prime, and let $SL(2, p)$ be the group of all 2 by 2 matrices of determinant 1 over the field with p elements. Show that $SL(2, p)$ has $p + 2$ conjugacy classes.

5. Let $F \subseteq K$ be a Galois extension. Define

$$\begin{array}{rccc} N: & K & \longrightarrow & F \\ & t & \mapsto & \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(t). \end{array}$$

Determine with justification whether each of the following three statements is true or false.

   (a) The function N is a well-defined multiplicative function.

   (b) If $G \subseteq \mathrm{GL}_n(K)$ is a subgroup of the general linear group and $\sigma \in \mathrm{Gal}(K/F)$, then the function $\sigma : G \to G$ given by $\sigma(a_{ij}) = (\sigma(a_{ij}))$ is a group homomorphism.

   (c) If $G \subseteq \mathrm{GL}_n(F)$ is a subgroup of the general linear group, then the function $N : G \to \mathrm{GL}_n(F)$ given by $N(a_{ij}) = (N(a_{ij}))$ is a group homomorphism.

6. Let $E = \mathrm{GF}(2^5)$ be the field with 32 elements, let $F = \mathrm{GF}(2)$ be the prime subfield of E, let A be an algebraic closure of E, and let c be a root in A of $f(x) = x^4 + x^3 + 1 \in F[x]$.

   (a) Show that $f(x)$ is irreducible in $F[x]$.

   (b) Find the splitting field of $f(x)$, regarded as a polynomial in $E[x]$.