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ALGEBRA

Ph.D. Preliminary Examination

August 14, 2009

Instructions:

(1) Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.

(2) Label each answer sheet with the problem number.

(3) Put your number, not your name, in the upper right hand corner of each page. If you have not received a number, please choose one (for example 1234) and notify the graduate secretary as to which number you have chosen.

(4) List below three faculty who could be contacted if there is some question about the results of this exam.

1. Give examples of the following or explain why no such example exists.
(a) A non-abelian group of order 48.
(b) A finite nilpotent group G and a normal subgroup N such that G/N is not nilpotent.
(c) A group G and a prime p such that G has exactly 5 Sylow p-subgroups.

2. If G is an abelian group acting on a finite set X, then the action of G on $X \times X$ defined by

$$g \cdot (x, y) = (g \cdot x, g \cdot y) \qquad \text{for all } (x, y) \in X \times X \text{ and } g \in G$$

has at least $|X|$ orbits.
(a) Prove this statement in the case that the action of G on X is transitive.
(b) Prove this statement in the case that the action of G on X is an arbitrary action.

3. Let R be a commutative unital ring with $1 \neq 0$. Show that if every proper principal ideal of R is a prime ideal, then R is a field.

4. Let $\mathrm{GL}_n(\mathbb{F}_q)$ be the general linear group over a finite field $\mathbb{F}_q$ with q elements and multiplicative group $\mathbb{F}_q^{\times}$.
(a) Show that for each $1 \leq m \leq n$, there exists an injective homomorphism $\mathbb{F}_{q^m}^{\times} \to \mathrm{GL}_n(\mathbb{F}_q)$.
(b) Give explicitly a matrix that generates a copy of $\mathbb{F}_9^{\times}$ inside $\mathrm{GL}_2(\mathbb{F}_3)$.
Hint: For (a) consider left multiplication by elements of $\mathbb{F}_{q^m}^{\times}$.

5. Let p be a prime number, q a power of p, and let f be an irreducible polynomial in $\mathbb{F}_p[x]$. Prove that any two irreducible factors of f over the field $\mathbb{F}_q$ have the same degree.

6. Let E be a finite Galois extension of F, and suppose that E has a subfield M such that $F \subsetneq M \subsetneq E$ and M is contained in every intermediate field between F and E that is different from F. Prove that
(a) $[E : F]$ is a prime power, and
(b) for any two intermediate fields K_1, K_2 between F and E we have $K_1 \leq K_2$ or $K_2 \leq K_1$.
Hint: Apply the Fundamental Theorem of Galois Theory.