Solutions

MATH 2400: CALCULUS 3

5:15 - 6:45 pm, Mon. Mar. 7, 2016

MIDTERM 2

I have neither given nor received aid on this exam.

Name:____________________________________

Check one below !

- ◯ **001** Watts ....(9AM)
- ◯ **002** Green .... (10AM)
- ◯ **003** Blakestad .... (11AM)
- ◯ **004** Mishev .... (12PM)
- ◯ **005** Washabaugh .... (1PM)
- ◯ **006** Bulin .... (2PM)
- ◯ **007** Chhay ....(3PM)

Notes, electronic devices, and any other aids are **not** permitted on this exam.

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **logical**, **legible**, and **correct**. Show all of your work, and give adequate explanations. No shown work even with the correct final answer will lead to no points. Only give one answer to each problem! If there are two different answers to one problem, the lower score will be chosen.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	16 pts	
2	17 pts	
3	17 pts	
4	17 pts	
5	16 pts	
6	17 pts	
TOTAL	100 pts	

1. **(16 points)** Find a parametrization of the surface given by the intersection of the plane $x + 2y + 3z = 12$ and the solid cylinder $x^2 + y^2 \leq 1$.

Since $x^2+y^2 \leq 1$, we let $x = r\cos\theta$, $y = r\sin\theta$.

Since $x+2y+3z=12$, we have

$$z = \frac{12-x-2y}{3} = 4 - \frac{1}{3}x - \frac{2}{3}y = 4 - \frac{r\cos\theta}{3} - \frac{2r\sin\theta}{3}.$$

Therefore, a parameterization of our surface is

$$x = r\cos\theta,\ y = r\sin\theta,\ z = 4 - \frac{r\cos\theta}{3} - \frac{2r\sin\theta}{3},$$

$$0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi.$$

2. **(17 points)** Consider the function $f(x, y) = (x^2 + y^2)^{\frac{3}{2}}$ and the point $P = \left(\sqrt{\frac{2}{3}}, 0\right)$.

(a) **(5 points)** Find the directional derivative of $f(x, y)$ at P in the direction towards the origin.

A vector from P in the direction towards the origin is $\vec{v} = \langle -\sqrt{\frac{2}{3}}, 0 \rangle$. A unit vector in the same direction is $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle -1, 0 \rangle$.

We have $\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 3x\sqrt{x^2+y^2}, 3y\sqrt{x^2+y^2} \rangle$

Hence, $D_{\vec{u}} f(P) = \vec{\nabla} f(P) \cdot \vec{u}$

$$= \langle 2, 0 \rangle \cdot \langle -1, 0 \rangle = -2$$

(b) **(4 points)** In what (unit) direction does $f(x, y)$ have its maximum rate of change at P?

The unit direction of maximum rate of change of f at P is given by

$$\frac{\vec{\nabla} f(P)}{\|\vec{\nabla} f(P)\|} = \frac{\langle 2, 0 \rangle}{\|\langle 2, 0 \rangle\|} = \langle 1, 0 \rangle$$

(c) **(4 points)** What is the maximum rate of change in the direction from part (b)?

$$\|\vec{\nabla} f(P)\| = \|\langle 2,0\rangle\| = 2$$

(d) **(4 points)** Find and sketch the set of all points Q at which the maximum rate of change of $f(x, y)$ is equal to the maximum rate of change at P from part (c).

Let $Q=(x,y)$. Then we need

$$\|\vec{\nabla} f(Q)\| = 2$$

$$\|\vec{\nabla} f(x,y)\| = 2$$

$$\|\langle 3x\sqrt{x^2+y^2},\, 3y\sqrt{x^2+y^2}\rangle\| = 2$$

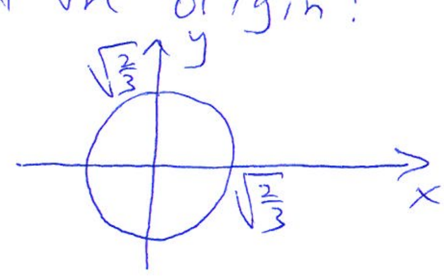
$$\sqrt{9x^2(x^2+y^2)+9y^2(x^2+y^2)} = 2$$

$$\sqrt{9(x^2+y^2)^2} = 2$$

$$3(x^2+y^2) = 2$$

$$x^2+y^2 = \frac{2}{3} = \left(\sqrt{\frac{2}{3}}\right)^2$$

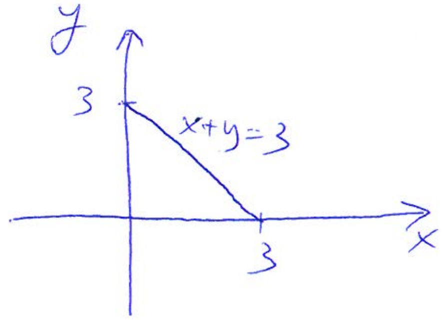
This is a circle of radius $\sqrt{\frac{2}{3}}$ centered at the origin:



3. (17 points) Find the absolute maximum and absolute minimum values of

$$f(x, y) = xy - 8x - y^2 + y + 2$$

over the (closed) triangular region with vertices $(0, 0)$, $(3, 0)$, and $(0, 3)$.



$f_x = y - 8 = 0 \Rightarrow y = 8$

$f_y = x - 2y + 1 = 0 \Rightarrow x = 2y - 1 = 2(8) - 1 = 15$

However, the point $(15, 8)$ is not in our triangular region (as $15 + 8 = 23 > 3$), so we will ignore it.

We will consider the 3 corner boundary points $(0,0)$, $(3,0)$, $(0,3)$.

Next:

On the boundary $y = 0$, $0 \le x \le 3$: $f(x,0) = -8x + 2 = f_1(x)$

$f_1'(x) = -8 \neq 0$ for any x, so no critical points.

On the boundary $x = 0$, $0 \le y \le 3$: $f(0,y) = -y^2 + y + 2 = f_2(y)$

$f_2'(y) = -2y + 1 = 0 \iff y = \frac{1}{2}$. A critical point is $(0, \frac{1}{2})$

On the boundary $x + y = 3$, we have $x = 3 - y$, and so

$f(x,y) = (3-y)y - 8(3-y) - y^2 + y + 2 = -2y^2 + 12y - 22 = f_3(y)$

$f_3'(y) = -4y + 12 = 0 \iff y = 3$. This leads to the corner point $(0,3)$.

We have

(x,y)	$f(x,y)$
$(0,0)$	2
$(3,0)$	-22
$(0,3)$	-4
$(0,\frac{1}{2})$	$9/4$

Hence, the absolute maximum value of f is $\frac{9}{4}$ at the point $(0, \frac{1}{2})$, and the absolute minimum value of f is -22 at the point $(3,0)$.

4. (17 points) Find all points on the hyperboloid of one sheet $x^2+y^2-z^2=1$ where the tangent plane is parallel to the plane $x+y-z=0$.

Let $F(x,y,z)=x^2+y^2-z^2$. The hyperboloid of one sheet $x^2+y^2-z^2=1$ is the level surface $F(x,y,z)=1$.
A normal vector to our hyperboloid of one sheet at the point (x,y,z) is given by

$$\vec{\nabla}F(x,y,z)=\langle F_x,F_y,F_z\rangle=\langle 2x,2y,-2z\rangle.$$

A normal vector to the plane $x+y-z=0$ is
$\vec{n}=\langle 1,1,-1\rangle$.

We need $\vec{\nabla}F(x,y,z)$ and $\vec{n}$ to be parallel, i.e.

$$\vec{\nabla}F(x,y,z)=\lambda\vec{n} \text{ for some } \lambda\neq 0.$$

$$\langle 2x,2y,-2z\rangle=\lambda\langle 1,1,-1\rangle$$

$$2x=\lambda,\ 2y=\lambda,\ 2z=\lambda$$

$$x=\frac{\lambda}{2},\ y=\frac{\lambda}{2},\ z=\frac{\lambda}{2}$$

Since the point (x,y,z) lies on $x^2+y^2-z^2=1$, we have

$$\left(\frac{\lambda}{2}\right)^2+\left(\frac{\lambda}{2}\right)^2-\left(\frac{\lambda}{2}\right)^2=1 \iff \left(\frac{\lambda}{2}\right)^2=1 \iff \lambda^2=4 \iff \lambda=2 \text{ or } \lambda=-2$$

Since our points (x,y,z) are given by $\left(\frac{\lambda}{2},\frac{\lambda}{2},\frac{\lambda}{2}\right)$, we have 2 such points: $(1,1,1)$ and $(-1,-1,-1)$.

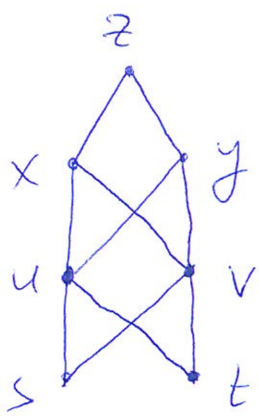
5. (16 points) Let

$$z = f(x,y), \quad x = u^2v^2 + 3, \quad y = -\cos(u) - v, \quad u = \frac{s}{t}, \quad v = e^{st}.$$

Suppose that f is a differentiable function of x and y and that

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + 2xy + y^2 + 1} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{x^2 + 2xy + y^2 + 1}.$$

Find $\left.\dfrac{\partial z}{\partial s}\right|_{(s,t)=(0,1)}$



By the Chain Rule,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{\partial v}{\partial s}$$

We compute $\dfrac{\partial x}{\partial u} = 2uv^2$, $\dfrac{\partial x}{\partial v} = 2u^2v$, $\dfrac{\partial y}{\partial u} = \sin u$, $\dfrac{\partial y}{\partial v} = -1$, $\dfrac{\partial u}{\partial s} = \dfrac{1}{t}$, $\dfrac{\partial v}{\partial s} = te^{st}$

We have $u(0,1) = 0$ and $v(0,1) = 1$. From here, we note that

$$\left.\frac{\partial x}{\partial u}\right|_{(s,t)=(0,1)} = \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(0,1)} = 0, \quad \left.\frac{\partial x}{\partial v}\right|_{(s,t)=(0,1)} = \left.\frac{\partial x}{\partial v}\right|_{(u,v)=(0,1)} = 0,$$

and $\left.\dfrac{\partial y}{\partial u}\right|_{(s,t)=(0,1)} = \left.\dfrac{\partial y}{\partial u}\right|_{(u,v)=(0,1)} = \sin(0) = 0$. Hence

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(0,1)} = \left(\left.\frac{\partial z}{\partial y}\right|_{(s,t)=(0,1)}\right)\left(\left.\frac{\partial y}{\partial v}\right|_{(s,t)=(0,1)}\right)\left(\left.\frac{\partial v}{\partial s}\right|_{(s,t)=(0,1)}\right)$$

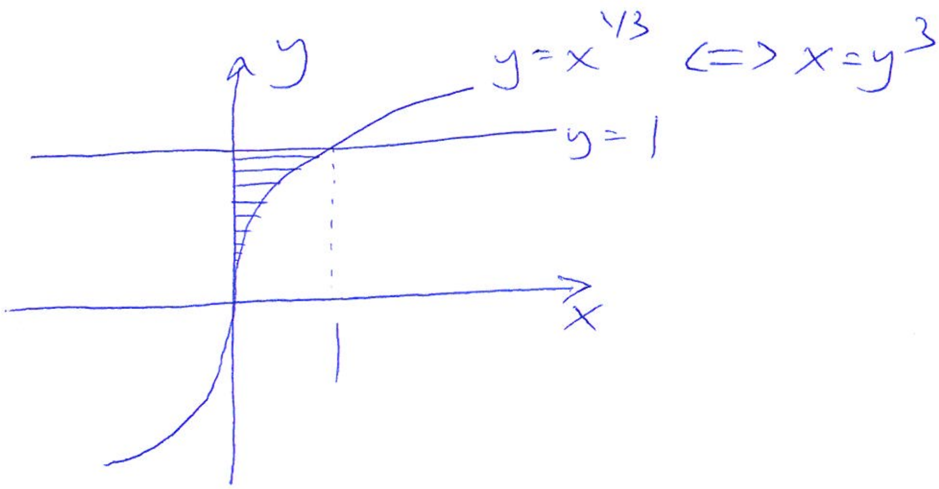
We have $\left.\dfrac{\partial v}{\partial s}\right|_{(s,t)=(0,1)} = 1$ and $\left.\dfrac{\partial y}{\partial v}\right|_{(s,t)=(0,1)} = \left.\dfrac{\partial y}{\partial v}\right|_{(u,v)=(0,1)} = -1$.

Finally, when $(u,v) = (0,1)$, $x(0,1) = 3$ and $y(0,1) = -2$. Thus

$\left.\dfrac{\partial z}{\partial y}\right|_{(s,t)=(0,1)} = \left.\dfrac{\partial z}{\partial y}\right|_{(x,y)=(3,-2)} = \dfrac{1}{2}$. We obtain $\left.\dfrac{\partial z}{\partial s}\right|_{(s,t)=(0,1)} = \left(\dfrac{1}{2}\right)(-1)(1) = -\dfrac{1}{2}$

6. (17 points) Compute the iterated integral

$$\int_0^1 \int_{x^{1/3}}^1 e^{y^4}\, dy\, dx.$$



We reverse the order of integration:

$$\int_0^1 \int_{x^{1/3}}^1 e^{y^4}\, dy\, dx = \int_0^1 \int_0^{y^3} e^{y^4}\, dx\, dy$$

$$= \int_0^1 \left(x e^{y^4} \Big|_{x=0}^{y^3} \right) dy = \int_0^1 y^3 e^{y^4}\, dy$$

$$= \frac{e^{y^4}}{4} \Big|_0^1 = \frac{e-1}{4}$$