MATH 2400: Calculus 3, Fall 2014
Midterm 1

September 17, 2014

NAME AND SIGNATURE:

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

Circle your section.

001 G. Rea .... (9am)

002 J. Migler .... (10am)

003 I. Mishev .... (11am)

004 M. Roy .... (12pm)

005 T. Davison .... (1pm)

006 C. Farsi .... (2pm)

007 D. Monk .... (3pm)

008 K. Havasi .... (9am)

009 J. Nishikawa .... (11am)

You must show all of your work. Please write legibly and box your answers. The use of calculators, books, notes, etc. is not permitted on this exam. Please provide exact answers when possible. For example, if the answer is π, write the symbol "π" and not the decimal 3.14159. . . .

Question	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) (a) Find the area of the parallelogram determined by the following vectors: $\vec{v}_1 = \langle 2, 3, -3\rangle$, $\vec{v}_2 = \langle 1, 5, 0\rangle$

Solution: The area of the parallelogram is given by

$$\begin{aligned}
|\vec{v}_1 \times \vec{v}_2| &= \left| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -3 \\ 1 & 5 & 0 \end{pmatrix} \right| \\
&= \left| 15\vec{i} - 3\vec{j} + 7\vec{k} \right| \\
&= \sqrt{15^2 + 3^2 + 7^2} \\
&= \sqrt{283}
\end{aligned}$$

(b) Using vectors, find the angle between the two planes below. Leave your answer in terms of inverse trigonometric functions.

$$3x + 6y + z = 5 \quad \text{and} \quad 2x - y + \frac{1}{2}z = -7$$

Solution: These planes have normal vectors $\vec{n}_1 = \langle 3, 6, 1\rangle$ and $\vec{n}_2 = \langle 2, -1, \frac{1}{2}\rangle$. The angle θ between them can be found from the equation $\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2| \cos\theta$.

$$\begin{aligned}
|\vec{n}_1| &= \sqrt{46} \\
|\vec{n}_2| &= \sqrt{\frac{11}{2}} \\
\vec{n}_1 \cdot \vec{n}_2 &= 3 \cdot 2 + 6 \cdot -1 + 1 \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right) \\
&= \cos^{-1}\left(\frac{1}{2\sqrt{253}}\right)
\end{aligned}$$

(c) Using vector products, find the volume of the parallelepiped determined by the following vectors: $\vec{v}_1 = \langle 2, 2, -3\rangle$, $\vec{v}_2 = \langle 0, 2, -1\rangle$, $\vec{v}_3 = \langle -3, 2, -1\rangle$.

Solution: The volume of the parallelepiped is given by

$$\begin{aligned}
|\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)| &= |\vec{v}_1 \cdot \langle 0, 3, 6\rangle| \\
&= |-12| \\
&= 12
\end{aligned}$$

2. (25 points) (a) Find an equation of the line passing through the point $P_0(1, 2, -3)$ and parallel to the vector $\langle 1, -1, 2\rangle$.

Solution: The line can be expressed parametrically by

$$\vec{r}(t) = \vec{P}_0 + t\vec{v} = \langle 1 + t, 2 - t, -3 + 2t\rangle$$

or equivalently,

$$\begin{aligned} x &= 1 + t \\ y &= 2 - t \\ z &= -3 + 2t \end{aligned}$$

By solving for t we can also express this line symmetrically as

$$x - 1 = -y + 2 = \frac{z + 3}{2}$$

(b) Find an equation of the plane passing through $P_0(1, 2, -3)$, $P_1(0, 2, 1)$, and $P_2(-1, 5, 2)$.

Solution: Consider for example the vectors $\overrightarrow{P_0P_1} = \langle -1, 0, 4\rangle$ and $\overrightarrow{P_0P_2} = \langle -2, 3, 5\rangle$. Then

$$\begin{aligned} \vec{n} &= \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} \\ &= \langle -12, -3, -3\rangle \end{aligned}$$

is a normal vector to the plane. Choosing P_0 for example, the plane is given by the equation

$$\vec{n} \cdot (\langle x, y, z\rangle - P_0) = 0$$

Hence the equation of the plane is

$$4x + y + z = 3$$

(c) What is the distance of the point $P_3(-1,1,1)$ from the plane passing through $P_0(1,2,-3)$ with normal vector $\langle 1,0,-4\rangle$?

Solution: We project the displacement vector $\overrightarrow{P_0P_3} = \langle -2,-1,4\rangle$ onto the normal vector $\vec{n} = \langle 1,0,-4\rangle$:

$$\begin{aligned}\frac{|\overrightarrow{P_0P_3}\cdot\vec{n}|}{|\vec{n}|} &= \frac{|-2+0-16|}{\sqrt{1^2+0^2+4^2}}\\ &= \frac{18}{\sqrt{17}}\end{aligned}$$

(d) Find an equation (or equations) representing all the lines passing through the point $P_0(1,2,-3)$ and perpendicular to the vector $\langle 1,-1,1\rangle$.

Solution: All such lines constitute the plane

$$\langle 1,-1,1\rangle\cdot(\langle x,y,z\rangle - \vec{P}_0) = 0$$

or equivalently

$$z = -4 - x + y$$

Hence, all such lines have parametric equations

$$\begin{aligned} x &= 1 + at\\ y &= 2 + bt\\ z &= -3 - at + bt\end{aligned}$$

for any real numbers a and b.

3. (25 points) Let $z = f(x, y) = 5 + \sqrt{x^2 + (y-3)^2 - 1}$.

   (a) Find the domain of the function $f(x, y) = 5 + \sqrt{x^2 + (y-3)^2 - 1}$.

   Solution: The domain is the set of all (x, y) values that may be used as inputs for f. Thus we need

   $$x^2 + (y-3)^2 - 1 \geq 0$$

   or equivalently

   $$x^2 + (y-3)^2 \geq 1$$

   This is the region outside and including the circle centered at $(0, 3)$ with radius 1.

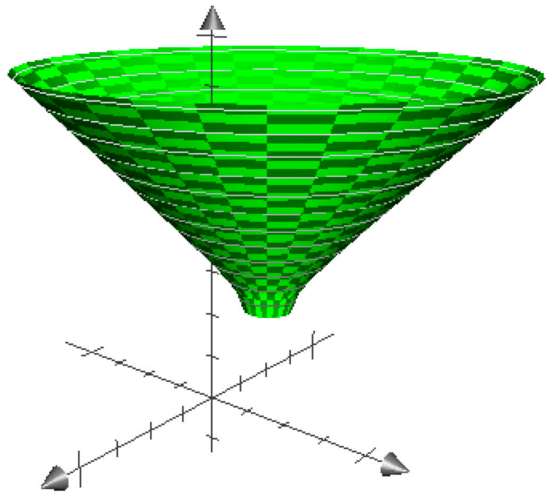
   (b) Write equations for the horizontal traces (or cross-sections) of the function $z = f(x, y) = 5 + \sqrt{x^2 + (y-3)^2 - 1}$ at $z = 5$, $z = 10$, and $z = 20$.

   Solution:

   $$\begin{aligned} z = 5 : &\quad x^2 + (y-3)^2 = 1 \\ z = 10 : &\quad x^2 + (y-3)^2 = 6 \\ z = 20 : &\quad x^2 + (y-3)^2 = 16 \end{aligned}$$

   (c) Graph the horizontal traces (or cross-sections) of the function $z = f(x, y) = 5 + \sqrt{x^2 + (y-3)^2 - 1}$ at $z = 5$, $z = 10$, and $z = 20$.

   Solution: Your graphs should be circles centered at $(0, 3)$ with radii 1, $\sqrt{6}$, and 4, respectively.



(d) By using traces, sketch the graph of the function $z = f(x, y) = 5 + \sqrt{x^2 + (y-3)^2 - 1}$ for $z \geq 5$.

4. (25 points) Consider the equations

   (a) $x^2 + y^2 - z = 0$ in Cartesian (= rectangular) coordinates.

   (b) $r^2 - z^2 = 0$ in cylindrical coordinates.

   (c) $\rho^2(1 + \cos\phi) = 1$ in spherical coordinates.

   (i) Which quadric surface, if any, does equation (a) represent?

   Solution: The graph of $z = x^2 + y^2$ is an elliptic (in fact, circular) paraboloid.

   (ii) Give an equation in cylindrical coordinates representing the surface given by (a).

   Solution: $r^2 - z = 0$.

   (iii) Give an equation in spherical coordinates representing the surface given by (a).

   Solution:

$$\begin{aligned} x^2 + y^2 - z &= 0 \\ x^2 + y^2 + z^2 - z^2 - z &= 0 \\ \rho^2 - \rho^2 \cos^2\phi - \rho\cos\phi &= 0 \end{aligned}$$

(iv) Give an equation in Cartesian (= rectangular) coordinates representing the surface given by (b).

Solution: $x^2 + y^2 - z^2 = 0$.

(v) Give an equation in Cartesian (= rectangular) coordinates representing the surface of equation (c).

Solution:

$$\begin{aligned} \rho^2(1 + \cos\phi) &= 1 \\ \rho^2 + \rho\cos\phi \cdot \rho &= 1 \\ x^2 + y^2 + z^2 + z\sqrt{x^2 + y^2 + z^2} &= 1 \end{aligned}$$