Solutions:

(1)

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{\frac{(x+h)^2}{y+1} - \frac{x^2}{y+1}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h(y+1)} = \lim_{h \to 0} \frac{h(2x+h)}{h(y+1)}$$

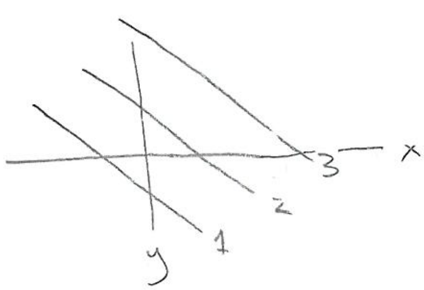
$$= \lim_{h \to 0} \frac{2x+h}{y+1} = \frac{2x}{y+1}$$

(2)

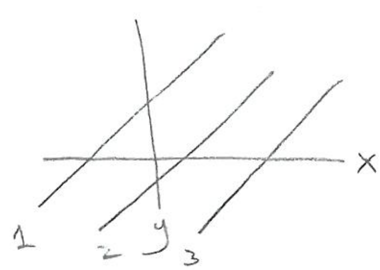
$$g_y(x,y) = \lim_{h \to 0} \frac{g(x,y+h) - g(x,y)}{h} = \lim_{h \to 0} \frac{e^{y+h}\cos(x) - e^y\cos(x)}{h} = \lim_{h \to 0} e^y\cos(x)\left(\frac{e^h - 1}{h}\right)$$

$$= e^y\cos(x) \lim_{h \to 0}\left(\frac{e^h-1}{h}\right) = e^y\cos(x)$$ (Use L'hôpital's Rule)

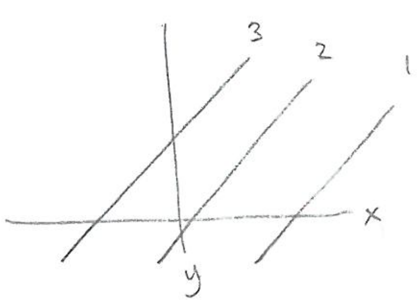
(3) (a)



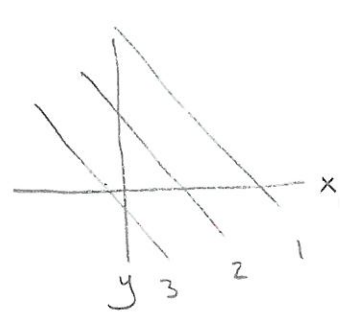
(b)



(c)



(d)



(4)

$$f_x(0,0) = \frac{d}{dx}\left(f(x,0)\right)\Big|_{x=0} = \frac{d}{dx}\left(\frac{x}{2}\right)\Big|_{x=0} = \frac{1}{2}\Big|_{x=0} = \frac{1}{2}$$

(5) (a) The tangent line to the graph of f at (a,b) is,

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

The tangent line to the contour of f through (a,b) will be,

$$f(a,b) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

which is,

$$0 = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(b) Solving for y,

$$-f_y(a,b)(y-b) = f_x(a,b)(x-a)$$

$$y = -\frac{f_x(a,b)}{f_y(a,b)}(x-a) + b$$

So slope is:

$$-\frac{f_x(a,b)}{f_y(a,b)}$$

(6)

$$T(2.04, 0.97) \approx$$

$$T(2,1) + T_x(2,1)(2.04-2) + T_y(2,1)(0.97-1)$$

$$= 135 + 16(.04) + (-15)(-.03)$$

(7) The tangent plane $L(x,y)$ at $(2,1,4)$ is:

$$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$f_x(x,y) = x \qquad f_y(x,y) = 4y$$

$$L(x,y) = 4 + 2(x-2) + 4(y-1)$$

(8) Since f is diff'able, ($\vec{u}$ is unit vector)

$$f_{\vec{u}}(x,y) = f_x(x,y)\, u_1 + f_y(x,y)\, u_2$$

$$= \nabla f(x,y) \cdot \vec{u}$$

$$= \|\nabla f(x,y)\|\, \|\vec{u}\| \cos\theta$$

$$= \|\nabla f(x,y)\| \cdot 1 \cdot \cos\theta$$

This last quantity is maximized when $\theta = 0$, that is when $\vec{u}$ points in the direction of $\nabla f(x,y)$. And, the maximum rate of change is $\|\nabla f(x,y)\|$.

(9) (a) The displacement vector from $(3,1)$ to $(1,2)$ is $\langle 2-1, 1-3\rangle = \langle 1,-2\rangle$.

$\|\langle 1,-2\rangle\| = \sqrt{1^2+(-2)^2} = \sqrt{5}$.

Then, the average rate of change is,

$$\frac{f(1,2) - f(3,1)}{\sqrt{5}} = \frac{1^2 + \ln(2) - (3^2 + \ln(1))}{\sqrt{5}}$$

(b) Need to compute,

$$f_{\vec{u}}(3,1)$$

where $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$

$$f_x(x,y) = 2x \qquad f_y(x,y) = \frac{1}{y}$$

So

$$f_{\vec{u}}(3,1) = f_x(3,1)\frac{1}{\sqrt{5}} + f_y(3,1)\left(\frac{-2}{\sqrt{5}}\right)$$

$$= 2(3)\left(\frac{1}{\sqrt{5}}\right) + \frac{1}{1}\left(\frac{-2}{\sqrt{5}}\right)$$

(10) The displacement vector from $(2,1)$ to $(1,3)$ is,

$\langle 3-1, 1-2 \rangle = \langle 2, -1 \rangle$. A unit vector in this direction is,

$$\left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle = \vec{u}$$

The displacement vector from $(2,1)$ to $(5,5)$ is

$\langle 5-1, 5-2 \rangle = \langle 4, 3 \rangle$. A unit vector in this direction is,

$$\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle = \vec{v}$$

Now,

$$f_{\vec{u}}(2,1) = f_x(2,1)\, u_1 + f_y(2,1)\, u_2$$

$$f_{\vec{v}}(2,1) = f_x(2,1)\, v_1 + f_y(2,1)\, v_2$$

So, $$-\frac{2}{\sqrt{5}} = f_x(2,1)\left(\frac{2}{\sqrt{5}}\right) + f_y(2,1)\left(-\frac{1}{\sqrt{5}}\right)$$

$$1 = f_x(2,1)\left(\frac{4}{5}\right) + f_y(2,1)\left(\frac{3}{5}\right)$$

Then, can solve this system of equations for,

$f_x(2,1)$ and $f_y(2,1)$

(11) (a) $f_{\vec{u}}(0,0) = 0$

(b) $f_x(0,0)\,\frac{1}{\sqrt{2}} + f_y(0,0)\,\frac{1}{\sqrt{2}} =$

$$(1)\,\frac{1}{\sqrt{2}} + (0)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

Notice, $\frac{1}{\sqrt{2}} \neq 0$. Hence, f is <u>not</u> diff'able at $(0,0)$.

(12) (a) Define $F(x,y,z) = x^2 + y^2 - xyz$

$$\nabla F = \langle 2x - yz,\ 2y - xz,\ -xy \rangle$$

$$\nabla F(2,3,1) = \langle 2(2) - 3(1),\ 2(3) - 2(1),\ -2(3) \rangle$$

$$= \langle 1, 4, -6 \rangle$$

Tangent plane:

$$(x-2)1 + (y-3)4 + (z-1)(-6) = 0$$

(b) $$z = \frac{7 - x^2 - y^2}{-xy} = \frac{-7}{xy} + \frac{x}{y} + \frac{y}{x}$$

$$\frac{\partial z}{\partial x} = \frac{-7}{y}\left(\frac{-1}{x^2}\right) + \frac{1}{y} - \frac{y}{x^2} \qquad \frac{\partial z}{\partial y} = \frac{-7}{x}\left(\frac{-1}{y^2}\right) - \frac{x}{y^2} + \frac{1}{x}$$

$$\frac{\partial z}{\partial x}(2,3) = -\frac{7}{3}\left(\frac{-1}{2^2}\right) + \frac{1}{3} - \frac{3}{2^2}$$

$$\frac{\partial z}{\partial y}(2,3) = -\frac{7}{2}\left(\frac{-1}{3^2}\right) - \frac{2}{3^2} + \frac{1}{2}$$

Then, tangent plane is:

$$1 + \frac{\partial z}{\partial x}(2,3)(x-2) + \frac{\partial z}{\partial y}(y-3)$$

(14) By chain rule,

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$8 = \frac{\partial f}{\partial x}(5) + \frac{\partial f}{\partial y}(6)$$

$$4 = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(2)$$

Solve for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(15) Consider

$$h(x(t), y(t))$$

where $x(t) = y(t) = t$. Then,

$$z(t) = h(x(t), y(t))$$

So $$\frac{dz}{dt} = \frac{dh}{dt} = \frac{\partial h}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial h}{\partial y} \frac{dy}{dt}$$

$$= \left.\frac{\partial h}{\partial x}\right|_{(x(t),y(t))} \cdot 1 + \left.\frac{\partial h}{\partial y}\right|_{(x(t),y(t))} \cdot 1 \qquad (*)$$

Now, $$\frac{\partial h}{\partial x} = f'(x)\, g(y)$$

$$\frac{\partial h}{\partial y} = f(x)\, g'(y)$$

So $$(*) = f'(x(t))\, g(y(t)) + f(x(t))\, g'(y(t))$$

$$= f'(t)\, g(t) + f(t)\, g'(t)$$

(16)

(a) $$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h} = 0$$

Similarly,

$$f_y(0,0) = 0$$

(b)

$$f_x(x,y) = \frac{(y^3 - 3x^2y)(x^2+y^2) - (xy^3 - x^3y)(2x)}{(x^2+y^2)^2}$$

if $(x,y) \neq (0,0)$

$$f_y(x,y) = \frac{(3xy^2 - x^3)(x^2+y^2) - (xy^3 - x^3y)(2y)}{(x^2+y^2)^2}$$

if $(x,y) \neq (0,0)$

(c)

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0, 0+h) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3 \cdot h^2}{h^4} - 0}{h} = \lim_{h \to 0} \frac{h^5}{h^5} = 1$$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h, 0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(-h^3)(h^2)}{h^4} - 0}{h} = -1$$

Notice that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

(17) Notice that if $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Next, if $y=2x$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(2x)}{x^2+(2x)^2} = \lim_{x\to 0} \frac{2x^2}{5x^2} = \frac{2}{5}$$

Hence, f is not continuous at $(0,0)$, so therefore not diff'able.

(18) $f_x(x,y) = 2x + A$

$f_y(x,y) = 2y$

We want the critical point to occur at $(1,0)$.

So $0 = 2(1) + A$

$0 = 2(0)$

$A = -2$. Then, since $f_{xx}(1,0) > 0$, and the discriminant is $f_{xx}(1,0)\,f_{yy}(1,0) - f_{xy}(1,0)^2 =$

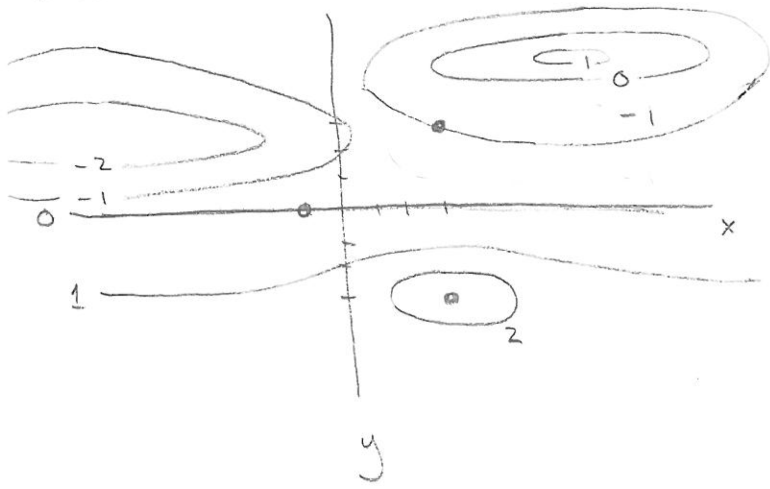
$2(2) - 0 > 0$, $(1,0)$ is a local minimum. Next

want

$20 = f(1,0)$

So $20 = 1^2 + (-2)(1) + 0 + B$ $\quad B = 21$

(19)



(20) $V = xyz$

$$3x + 2y + z = 1$$

$$z = 1 - 3x - 2y$$

$$V = xy(1 - 3x - 2y) = xy - 3x^2y - 2xy^2$$

$$\frac{\partial V}{\partial x} = y - 6xy - 2y^2 = 0$$

$$\frac{\partial V}{\partial y} = x - 3x^2 - 4xy = 0$$

$$y(1 - 6x - 2y) = 0$$

$$x(1 - 3x - 4y) = 0$$

$x = y = 0$ is minimum.

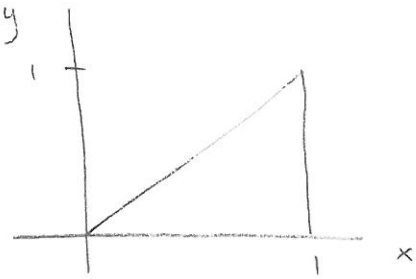
$$1 = 6x + 2y \qquad x = \frac{1}{9} \quad y = \frac{1}{6}$$

$$1 = 3x + 4y$$

$$z = 1 - 3\left(\frac{1}{9}\right) - 2\left(\frac{1}{6}\right)$$

(21)

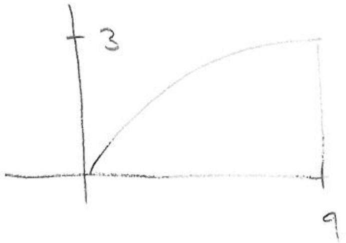
(a) $$\int_{y=0}^{y=1}\int_{x=y}^{x=1} e^{x^2}\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=x} e^{x^2}\,dy\,dx$$



$$= \int_0^1 e^{x^2} y \Big]_{y=0}^{y=x} dx = \int_0^1 e^{x^2} x\,dx =$$

$$\frac{1}{2} e^{x^2} \Big]_0^1 = \frac{1}{2}e^1 - \frac{1}{2}e^0 = \frac{1}{2}e - \frac{1}{2}.$$

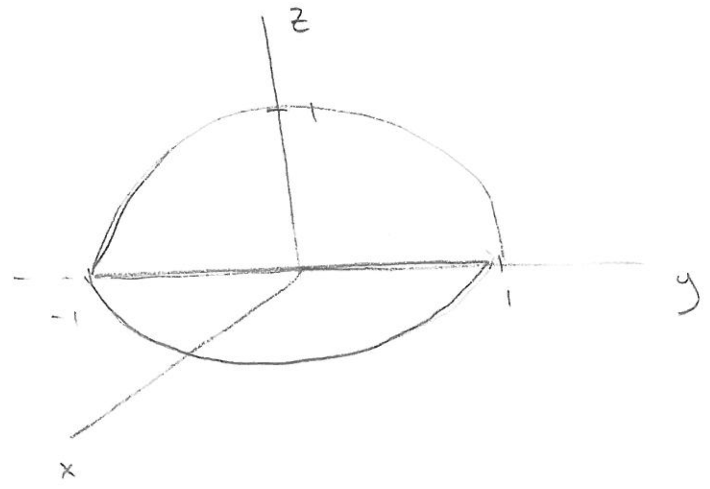
(b) $$\int_{y=0}^{y=3}\int_{x=y^2}^{x=9} y\sin(x^2)\,dx\,dy = \int_{x=0}^{x=9}\int_{y=0}^{y=\sqrt{x}} y\sin(x^2)\,dy\,dx$$



$$= \int_0^9 \frac{y^2 \sin(x^2)}{2} \Big]_0^{\sqrt{x}} dx$$

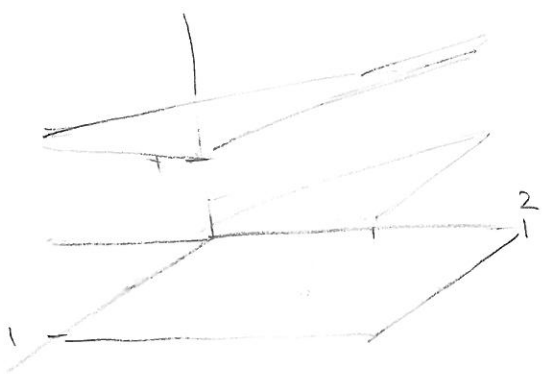
$$= \int_0^9 \frac{x\sin(x^2)}{2}\,dx = \frac{-\cos(x^2)}{4}\Big]_0^9 = \frac{-\cos(9)}{4} - - \frac{\cos(0)}{4}$$

(22)



Quarter of the unit sphere.

(23)



$$\int_0^1 \int_0^2 \int_{z=x+y}^{z=1+2x+2y} 1 \, dz\, dy\, dx$$