Solutions to Review 3:

(1) D is the disk $x^2+y^2 \le 1$. Hence, switch to polar. That is,

$$\iint_D \sqrt{1-x^2-y^2}\,dx\,dy = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} \sqrt{1-r^2}\, r\,dr\,d\theta \qquad (*)$$

Using u-sub with $u = 1-r^2$, we get that an anti-derivative of $(\sqrt{1-r^2})r$ with respect to r is,

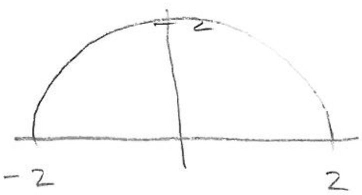
$$-\frac{1}{3}(1-r^2)^{3/2}$$

Hence,

$$(*) = \int_{\theta=0}^{\theta=2\pi} -\frac{1}{3}(1-r^2)^{3/2}\Big]_{r=0}^{r=1}\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1}{3}\,d\theta = \frac{1}{3}\theta\Big]_{\theta=0}^{\theta=2\pi} = \frac{2\pi}{3}$$

(2) Switch to polar again. The region of integration is drawn below:



$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2} (r\cos\theta)^2 (r\sin\theta)^2 \, r \, dr \, d\theta =$$

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2} r^5 \cos^2\theta \sin^2\theta \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \cos^2\theta \sin^2\theta \left(\frac{r^6}{6} \Big]_{r=0}^{r=2} \right) d\theta = \int_{\theta=0}^{\theta=\pi} \cos^2\theta \sin^2\theta \left(\frac{2^6}{6} - 0 \right) d\theta$$

$$= \frac{2^6}{6} \int_{\theta=0}^{\theta=\pi} \cos^2\theta \sin^2\theta \, d\theta \qquad (*)$$

Let's now find the indefinite integral,

$$\int \cos^2\theta \sin^2\theta \, d\theta$$

Note,

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \quad \text{and,}$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

Hence, $$\cos^2\theta\sin^2\theta = \left(\frac{1+\cos(2\theta)}{2}\right)\left(\frac{1-\cos(2\theta)}{2}\right)$$

$$= \frac{1-\cos^2(2\theta)}{4} = \frac{\sin^2(2\theta)}{4}$$

Using integration by parts,

$$\int \frac{\sin^2(2\theta)}{4}\,d\theta = \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{-\sin(2\theta)\cos(2\theta)+2\theta}{2}\right)$$

$$= \frac{1}{8}\left(\frac{-\sin(2\theta)\cos(2\theta)+2\theta}{2}\right)$$

Hence, $$(*) = \frac{2^6}{6}\left(\frac{1}{8}\right)\left[\frac{-\sin(2\theta)\cos(2\theta)}{2}+\theta\right]_{\theta=0}^{\theta=\pi}$$

$$= \frac{64}{48} (\pi) = \frac{4\pi}{3}$$

(3) To find the volume of the torus we use cylindrical coords:

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=\sin\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \sin\phi \left(\frac{\rho^3}{3} \Big]_{\rho=0}^{\rho=\sin\phi} \right) d\rho \, d\phi \, d\theta =$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \frac{\sin^4\phi}{3} \, d\phi \, d\theta \qquad (*)$$

Let's find the indefinite integral of,

$$\int \sin^4\phi \, d\phi$$

Using the identity,

$$\sin^2 \phi = \left(\frac{1 - \cos(2\phi)}{2} \right)$$

$$\sin^4 \phi = \left(\frac{1 - \cos(2\phi)}{2} \right)^2 = \frac{1}{4} \left(1 - 2\cos(2\phi) + \cos^2(2\phi) \right)$$

Integrating this, we get

$$\frac{1}{4} \left(\phi - 2\,\frac{\sin(2\phi)}{2} + \underbrace{\frac{\sin(2\phi)\cos(2\phi) + 2\phi}{4}}_{\text{via integration by parts.}} \right)$$

$$= \frac{\phi}{4} - \frac{\sin(2\phi)}{4} + \frac{\sin(2\phi)\cos(2\phi)}{4} + \frac{\phi}{8}$$

$$= \frac{3\phi}{8} - \frac{\sin(2\phi)}{4} + \frac{\sin(2\phi)\cos(2\phi)}{4}$$

$$(*) = \frac{1}{3} \int_{\theta=0}^{\theta=2\pi} \left. \frac{3\phi}{8} - \frac{\sin(2\phi)}{4} + \frac{\sin(2\phi)\cos(2\phi)}{4} \right]_{\phi=0}^{\phi=\pi} d\theta$$

$$= \frac{1}{3}\int_{\theta=0}^{\theta=2\pi} \frac{3\pi}{8}\,d\theta = \frac{\pi}{8}(2\pi) = \frac{\pi^2}{4}$$

(4) The charge of variables is given by:

$$x = \frac{1}{3}(u+v) \qquad y = \frac{1}{3}(v-2u)$$

When $y = x$:

$$\frac{1}{3}(v-2u) = \frac{1}{3}(u+v) \Rightarrow u = 0$$

When $y = -2x$:

$$\frac{1}{3}(v-2u) = -2\left(\frac{1}{3}(u+v)\right) \Rightarrow v = 0$$

When $y = x - 2$:

$$\frac{1}{3}(v-2u) = \frac{1}{3}(u+v) - 2 \Rightarrow u = 2$$

When $y = 3 - 2x$:

$$\frac{1}{3}(v-2u) = 3 - 2\left(\frac{1}{3}(u+v)\right) \Rightarrow v = 3$$

Also,

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{pmatrix} =$$

$$\frac{1}{9} - -\frac{2}{9} = \frac{1}{3}$$

So $$\iint_R (3x + 4y)\, dA = \int_{u=0}^{u=2} \int_{v=0}^{v=3} \left(3\left(\frac{1}{3}(u+v)\right)\right.$$

$$\left. + \; 4\left(\frac{1}{3}(v-2u)\right)\right)\left(\frac{1}{3}\right) dv\, du$$

$$= \int_{u=0}^{u=2} \int_{v=0}^{v=3} \left(u + v + \frac{4}{3}v - \frac{8}{3}u\right) \frac{1}{3}\, dv\, du$$

$$= \int_{u=0}^{u=2} \int_{v=0}^{v=3} \left(-\frac{5}{3}u + \frac{7}{3}v\right) \frac{1}{3}\, dv\, du = \frac{1}{3}\Bigg($$

$$\int_{u=0}^{u=2} -\frac{5}{3}uv + \frac{7}{6}v^2 \Bigg]_{v=0}^{v=3} du \Bigg)$$

$$= \frac{1}{3}\left(\int_{u=0}^{u=2} -5u + \frac{63}{6}\, du\right)$$

$$= \frac{1}{3}\left(-\frac{5u^2}{2} + \frac{63}{6}u\Big]_0^2\right)$$

$$= \frac{1}{3}\left(-\frac{5(2^2)}{2} + \frac{63(2)}{6}\right) = \frac{11}{3}$$

(5) Consider the vector field

$$\vec{G}(x,y) = \left\langle -\frac{1}{x^2}, -\frac{1}{2xy}\right\rangle \quad \text{for } x \neq 0 \text{ and}$$

$y \neq 0$. If $(x,y) \neq (0,0)$, then

$$\vec{G}(x,y) \cdot \vec{F}(x,y) = 0$$

We will find a parametrized curve $\vec{r}(t) = x(t)\vec{i}$ $+ y(t)\vec{j}$ thru $(1,2)$ such that

$$\frac{dx}{dt} = -\frac{1}{x^2} \quad \text{and} \quad \frac{dy}{dt} = -\frac{1}{2xy}$$

Solving $\frac{dx}{dt} = -\frac{1}{x^2}$ where $x(0) = 1$:

$$x^2 \, dx = -dt$$

$$\frac{x^3}{3} = -t + C$$

$$x^3 = -3t + C$$

$$x = (C - 3t)^{1/3}$$

$$1 = (C - 3(0))^{1/3}$$

$$C = 1$$

So $x(t) = (1 - 3t)^{1/3}$.

Solving $\frac{dy}{dt} = -\frac{1}{2xy}$ where $y(0) = 2$

$$y \, dy = -\frac{1}{2(1-3t)^{1/3}} \, dt$$

$$\frac{y^2}{2} = \frac{1}{4}(1-3t)^{2/3} + C$$

$$y = + \sqrt{\frac{1}{2}(1-3t)^{2/3} + C}$$

$$2 = \sqrt{\frac{1}{2}(1-3(0))^{2/3} + C}$$

$$4 = \frac{1}{2} + C \qquad C = \frac{7}{2}$$

So,

$$y(t) = \sqrt{\frac{1}{2}(1-3t)^{2/3} + \frac{7}{2}}$$

(6) $$\left.\frac{dx}{dt}\right|_{t=-2} = 3t^2 - 3 \Big|_{t=-2} = 9$$

$$\left.\frac{dy}{dt}\right|_{t=-2} = 2t - 2 \Big|_{t=-2} = -6$$

$$\left.\frac{dz}{dt}\right|_{t=-2} = 1$$

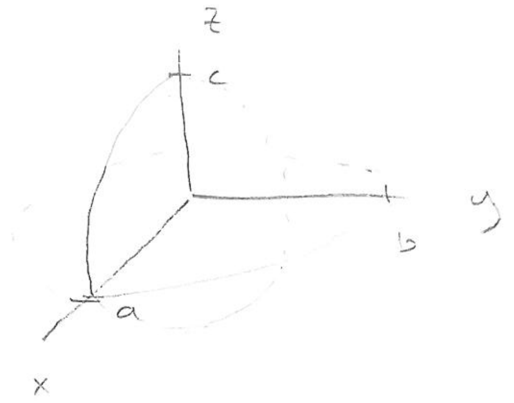
Equation for tangent line is

$$\vec{r}(-2) + t\langle 9, -6, 1\rangle =$$

$$\left((-2)^3 - 3(-2),\ (-2)^2 - 2(-2),\ -2+5\right) + t\langle 9, -6, 1\rangle$$

$= (-2, 8, 3) + t\langle 9, -6, 1\rangle$

(7) The surface we are parametrizing looks like



The sphere is parametrized by

$$x = \sin\phi \cos\theta$$

$$y = \sin\phi \sin\theta$$

$$z = \cos\phi$$

This surface is a slight modification

$$x = a \sin\phi \cos\theta$$

$$y = b \sin\phi \sin\theta$$

$$z = c \cos\phi$$

(8) The displacement vector from $(1,2,3)$ to $(3,5,7)$

is $\langle 3-1, 5-2, 7-3 \rangle = \langle 2, 3, 4 \rangle$.

Hence, a parametrization of the line is:

$$(1,2,3) + t\langle 2,3,4 \rangle =$$

$$\langle 1+2t,\ 2+3t,\ 3+4t \rangle$$

Let us minimize the squared distance to the origin, call it $h(t)$:

$$h(t) = (1+2t)^2 + (2+3t)^2 + (3+4t)^2$$

$$= 1 + 4t + 4t^2 + 4 + 12t + 9t^2 +$$

$$9 + 24t + 16t^2$$

$$= 14 + 40t + 29t^2$$

$$h'(t) = 58t + 40$$

$$0 = 58t + 40$$

$$t = \frac{-40}{58}$$ ← show a minimum

$\sqrt{h\left(-\frac{40}{52}\right)}$ is the shortest distance

(9)

(a) The change of variable from cartesian to cylindrical is:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Computing the Jacobian one will get:

r

(b) The change of variable from cartesian to spherical is:

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

The Jacobian is $\rho^2\sin\phi$

(c) Here is a picture:

$$
\begin{tikzcd}
\text{cylindrical} & \text{spherical} & \text{cartesian} \\
(r, \theta, z) \arrow[r, "\frac{\partial(\rho, \theta, \phi)}{\partial(r, \theta, z)} = ?"'] \arrow[rr, bend right, "\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r"'] & (\rho, \theta, \phi) \arrow[r, "\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 \sin\phi"'] & (x, y, z)
\end{tikzcd}
$$

We know from reciation worksheet,

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \cdot \frac{\partial(\rho,\theta,\phi)}{\partial(r,\theta,z)}$$

$$r = \rho^2 \sin\phi \cdot \frac{\partial(\rho,\theta,\phi)}{\partial(r,\theta,z)}$$

$$\frac{r}{\rho^2 \sin\phi} = \frac{\partial(\rho,\theta,\phi)}{\partial(r,\theta,z)} \longrightarrow$$

Solve for ρ and ϕ in terms of (r, θ, z).

Now, $\rho^2 = r^2 + z^2$ and

$$\sin\phi = \frac{r}{\sqrt{r^2+z^2}}$$

so

$$\frac{\partial(\rho, \theta, \phi)}{\partial(r, \theta, z)} = \frac{r}{(r^2+z^2)\dfrac{r}{\sqrt{r^2+z^2}}}$$

$$= \frac{r}{\sqrt{r^2+z^2}\; r} = \frac{1}{\sqrt{r^2+z^2}} = \frac{1}{\rho}$$

<u>Note</u>: We want to find

$$\frac{\partial(r, \theta, z)}{\partial(\rho, \theta, \phi)} = \frac{1}{\dfrac{\partial(\rho, \theta, \phi)}{\partial(r, \theta, z)}} = \frac{1}{\frac{1}{\rho}} = \rho$$

(10) Need to find the t's where,

$$-9(t^2) - 2(2t+1) - 10(1-t^2) = 0$$

$$-9t^2 - 4t - 2 - 10 + 10t^2 = 0$$

$$t^2 - 4t - 12 = 0$$

$$(t+2)(t-6) = 0$$

$$t = -2 \qquad t = 6$$

(11) (a) Parametrize the upper half of circle $x^2+y^2 = 2$ by, $x(t) = -2\cos(t)$ $y(t) = 2\sin t$, (oriented clockwise). Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=\pi} \langle -2\sin t, -2\cos t \rangle \cdot \langle 2\sin t, 2\cos t \rangle \, dt$$

$$= \int_{t=0}^{t=\pi} -4(\sin^2 t + \cos^2 t) \, dt$$

$$= -4t \Big]_0^{\pi} = -4\pi.$$

(b) Parametrize the line segment from $(-2,0)$ to $(2,0)$.

$$(1-t)(-2,0) + t(2,0) = (-2+2t+2t, 0)$$

$$= \langle -2+4t, 0 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=1} \langle 0, -2+4t \rangle \cdot \langle 4, 0 \rangle \, dt = 0$$

(c) $\vec{F}$ is path dependent.

(12) C is traced out between $t=0$ and $t=\pi$.

Hence,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=\pi} \langle \cos t, \sin^3 t, -\sin(t) \rangle \cdot \langle -\sin t,$$

$$\cos t, 1 \rangle \, dt = \int_{t=0}^{t=\pi} -\sin t \cos t + \sin^3 t \cos t - \sin t \, dt$$

$$= \frac{(\cos t)^2}{2} + \frac{(\sin t)^4}{4} + \cos t \Bigg]_0^{\pi}$$

$$= \frac{1}{2} + 0 + (-1) - \left(\frac{1}{2} + 0 + 1 \right) = -2$$

(13) The normal vectors of these planes are:

$$\langle 1, 1, 1 \rangle$$

$$\langle 1, -2, 3 \rangle$$

$$\det \begin{pmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix} = (3 - -2)\vec{i} - (3 - 1)\vec{j} +$$

$$(-2 - 1)\vec{k} = \langle 5, -2, -3 \rangle$$

Now, $\langle 5, -2, -3 \rangle$ is parallel to the line of intersection of the two planes.

Need to find a point on the line of intersect-ion.

Suppose $x = 0$, then

$$y + z = 3$$

$$-2y + 3z = 0$$

So $-2y + 3(3 - y) = 0$

$$-2y + 9 - 3y = 0$$

$$y = \frac{9}{5}$$

$$\frac{9}{5} + z = 3$$

$$z = \frac{6}{5}$$

Hence, line of intersection is:

$$\left(0, \frac{9}{5}, \frac{6}{5}\right) + t\langle 5, -2, -3\rangle$$

(14) The displacement vector from $(1, 2, 3)$ to $(-3, 5, -7)$ is

$$\langle -3-1, 5-2, -7-3\rangle =$$

$$\langle -4, 3, -10\rangle$$

The displacement vector from $(1, 2, 3)$ to $(8, 12, 0)$ is,

$$\langle 8-1, 12-2, 0-3\rangle =$$

$$\langle 7, 10, -3\rangle$$

Hence, the parametrization of the plane is,

$$(1, 2, 3) + s\langle -4, 3, -10\rangle + t\langle 7, 10, -3\rangle$$

(15) $f(x,y,z) = \frac{x^3}{3} + yx + g(y,z)$

$f(x,y,z) = xy + y\cos(z) + k(x,z)$

$f(x,y,z) = \frac{z^2}{2} + y\cos(z) + l(x,y)$

Hence, $f(x,y,z) = \frac{x^3}{3} + yx + y\cos(z) + \frac{z^2}{2}$