MATH 2400: Calculus III, Fall 2013
MIDTERM #3

November 13, 2013

YOUR NAME:

Circle Your CORRECT Section

001 E. Angel . . . . (9am)
002 E. Angel . . . . (10am)
003 A. Nita . . . . (11am)
004 K. Selker . . . . (12pm)
005 I. Mishev . . . . (1pm)
006 C. Farsi . . . . (2pm)
007 R. Rosenbaum . . . . (3pm)
008 S. Henry . . . . (9am)

Important note: SHOW ALL WORK. BOX YOUR ANSWERS. Calculators are not allowed. No books, notes, etc. Throughout this exam, please provide exact answers where possible. That is: if the answer is 1/2, do not write 0.499 or something of that sort; if the answer is π, do not write 3.14159.

Problem	**Points**	**Score**
1	25	
2	25	
3	25	
4	25	
TOTAL	100	

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

SIGNATURE:

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1. (25 points) Let W be the three dimensional object that is bounded below by the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$ and bounded above by the sphere $x^2+y^2+z^2 = 9$. Suppose the density of W is distributed by

$$\delta(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}.$$

Find the mass of W.

The mass of W is given by the triple integral over W of the density. This is best evaluated using spherical coordinates.

In spherical coordinates, $\delta(\rho,\phi,\theta) = e^{(\rho^2)^{3/2}} = e^{\rho^3}$. The upper bounding sphere has equation $\rho = 3$ and the lower bounding cone has equation

$$z = \frac{r}{\sqrt{3}} \;\Rightarrow\; \frac{r}{z} = \sqrt{3} \;\Rightarrow\; \tan\phi = \sqrt{3} \;\Rightarrow\; \phi = \frac{\pi}{3}.$$

$$\begin{aligned}
\int_W \delta\, dV &= \int_0^{2\pi}\int_0^{\pi/3}\int_0^3 e^{\rho^3}\,\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3} \left.\frac{e^{\rho^3}}{3}\right|_{\rho=0}^{3} \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3} \left.\left(\frac{e^{27}-1}{3}\right)\right|_{\rho=0}^{3} \sin\phi\, d\phi\, d\theta \\
&= \left(\frac{e^{27}-1}{3}\right) \left.(-\cos\phi)\right|_{\phi=0}^{\pi/3} (2\pi) \\
&= \frac{\pi}{3}\left(e^{27}-1\right).
\end{aligned}$$

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2. (25 points) Use the change of variables $s = y$, $t = y - x^3$ to evaluate $\int\int_R x^2\,dx\,dy$ over the region R bounded by $y = 2$, $y = 8$, $y = x^3$, and $y = x^3 + 8$.

We will use the change of variables formula

$$\int_R f(x,y)\,dx\,dy = \int_T f(x(s,t), y(s,t)) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt.$$

Since $x^3 = y - t$, the transformation is given by

$$\begin{cases} x(s,t) &= (s-t)^{1/3} \\ y(s,t) &= s \end{cases}$$

The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{1}{3}(s-t)^{-2/3} & -\frac{1}{3}(s-t)^{-2/3} \\ 1 & 0 \end{vmatrix} = \frac{1}{3}(s-t)^{-2/3}.$$

The boundary of the region T in the st-plane corresponding to R is given by

xy-plane	st-plane
$y = 2$	$s = 2$
$y = 8$	$s = 8$
$y = x^3$	$t = 0$
$y = x^3 + 8$	$t = 8$

Finally, the integrand can be written in terms of s and t as

$$f(x(s,t), y(s,t)) = (x(s,t))^2 = (s-t)^{2/3},$$

so that the change of variables formula gives

$$\begin{aligned} \int_R f(x,y)\,dx\,dy &= \int_T f(x(s,t), y(s,t)) \left|\frac{\partial(x,y)}{\partial(s,t)}\right| ds\,dt \\ &= \int_2^8 \int_0^8 (s-t)^{2/3} \left(\frac{1}{3}(s-t)^{-2/3}\right) ds\,dt \\ &= \frac{1}{3}(8)(6) = 16. \end{aligned}$$

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3. Let

$$\vec{F}(x,y) = (3x)\,\vec{i} + (-3y)\,\vec{j}\,.$$

(a) **(15 points)** Find a parametrization for the flow line of $\vec{F}$ that passes through the point (1,3).

We use the definition of a flow line, that is, $\vec{r}(t)$ parameterizes a flow line if

$$\vec{F}(\vec{r}(t)) = \vec{r}\,'(t)$$

which we can rewrite as

$$\begin{aligned} \vec{F}(x(t), y(t)) &= x'(t)\vec{i} + y'(t)\vec{j} \\ 3(x(t))\vec{i} - 3(y(t))\vec{j} &= x'(t)\vec{i} + y'(t)\vec{j}. \end{aligned}$$

The components of these two vectors are equal, giving the system of differential equations

$$\begin{cases} x' = 3x \\ y' = -3y \end{cases} \qquad \Rightarrow \qquad \begin{cases} x(t) = C_1 e^{3t} \\ y(t) = C_2 e^{-3t} \end{cases}$$

Using the initial conditions $x(0) = 1$ and $y(0) = 3$ we obtain the parameterization

$$\begin{cases} x(t) = e^{3t} \\ y(t) = 3e^{-3t} \end{cases} \qquad \Rightarrow \qquad \vec{r}(t) = e^{3t}\vec{i} + 3e^{-3t}\vec{j}$$

(b) **(10 points)** Write Cartesian equations for the flow lines of the vector field $\vec{F}(x,y)$. Cartesian equations are equations involving the standard calculus variables x and y, as well as constants.

We can rewrite $y(t)$ in the general solution as

$$y(t) = C_2(e^{3t})^{-1} = C_2\left(\frac{x(t)}{C_1}\right)^{-1}$$

so that the Cartesian equation describing the curve is $y = \dfrac{C_2}{C_1}\dfrac{1}{x}$ or combining the two constants into a single constant C, $xy = C$.

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4. Consider the vector field

$$\vec{F}(x,y) = 4\left[(1+x^2y^2)xy^2\right]\vec{i} + 4\left[(1+x^2y^2)x^2y+1\right]\vec{j}$$

(a) **(10 points)** Let C be the segment of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $(2,1/2)$. Evaluate the line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

by parametrizing C. Do **not** use the Fundamental Theorem of Calculus for Line Integrals.

Parameterize C by $\vec{r}(t) = t\vec{i} + \frac{1}{t}\vec{j}$ where $1 \le t \le 2$. Then $\vec{F}$ evaluated along C is

$$\begin{aligned}\vec{F}(x(t),y(t)) &= 4\left[(1+(t)^2(1/t)^2)(t)(1/t)^2\right]\vec{i} + 4\left[(1+(t^2)(1/t^2))(t)^2(1/t)+1\right]\vec{j}\\ &= 4\left(\frac{2}{t}\vec{i} + (2t+1)\vec{j}\right)\end{aligned}$$

The line integral is

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_1^2 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt\\ &= \int_1^2 4\left(\frac{2}{t}\vec{i} + (2t+1)\vec{j}\right)\cdot\left(\vec{i} - \frac{1}{t^2}\vec{j}\right)\,dt\\ &= \int_1^2 4\left(\frac{2}{t} - \frac{2t+1}{t^2}\right)\,dt\\ &= \int_1^2 \left(-\frac{4}{t^2}\right)\,dt\\ &= \left.\frac{4}{t}\right|_1^2 = -2.\end{aligned}$$

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(b) **(10 points)** Show that $\vec{F}$ is a conservative vector field by finding a potential function f for $\vec{F}$.

$$f(x,y) = \underline{\hspace{6cm}}.$$

If f is a potential function for $\vec{F}$ then $\operatorname{grad} f = \vec{F}$ by definition. Comparing the first components of these vectors, $\partial f/\partial x = 4\left[(1+x^2y^2)xy^2\right]$ implies

$$f(x,y) = 4(1+x^2y^2)^2 + g(y)$$

Differentiating with respect to y, we see

$$\partial f/\partial y = 4\left[(1+x^2y^2)x^2y\right] + g'(y)$$

which means $g'(y) = 4$ upon comparing the second components of $\operatorname{grad} f = \vec{F}$. Hence

$$f(x,y) = 4(1+x^2y^2)^2 + 4y$$

is a potential function for $\vec{F}$.

(c) **(5 points)** Use the Fundamental Theorem of Calculus for Line Integrals to evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the segment of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $(2,1/2)$ given in part (a).

By the Fundamental Theorem of Calculus for Line Integrals,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_C (\operatorname{grad} f) \cdot d\vec{r} \\
&= f(2,1/2) - f(1,1) \\
&= 4(1+(2)^2(1/2)^2)^2 + 4(1/2) - (4(1+(1)^2(1)^2)^2 + 4(1)) \\
&= -2
\end{aligned}$$