Math 2400: Review Exam 2

1. Use the limit definition of the partial derivative to compute the partial derivative function, $f_x(x, y)$, for the function $f(x, y) = \dfrac{x^2}{y+1}$.

2. Use the limit definition of the partial derivative to compute the partial derivative function, $g_y(x, y)$, for the function $g(x, y) = e^y \cos(x)$.

3. In each case, give a possible contour diagram for the function $f(x, y)$ if:
   (a) $f_x > 0$ and $f_y > 0$
   (b) $f_x > 0$ and $f_y < 0$
   (c) $f_x < 0$ and $f_y > 0$
   (d) $f_x < 0$ and $f_y < 0$

4. Compute $f_x(0, 0)$ for the function,

$$f(x, y) = \frac{y \sin(e^{x^2}) \ln(\sqrt{x+2}) 2^x + x}{y^2 \cos(x^{\frac{3}{2}}) + 2}.$$

   Note: There is a fast way to do this.

5. Let $f(x, y)$ be differentiable at $(a, b)$.
   (a) If $f_x(a, b)$ or $f_y(a, b)$ is non-zero, show that that an equation of the line tangent to the contour of $f$ through $(a, b)$ is $f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0$.
   (b) Find the slope of the tangent line if $f_y(a, b) \neq 0$.

6. An unevenly heated plate has temperature $T(x, y)$ in degrees celsius at the point $(x, y)$. If $T(2, 1) = 135$, $T_x(2, 1) = 16$, and $T_y(2, 1) = -15$, estimate the temperature at the point $(2.04, 0.97)$.

7. Find the equation of the tangent plane for the function $f(x, y) = \frac{1}{2}(x^2 + 4y^2)$ at the point $(2, 1, 4)$.

8. Suppose that $f(x, y)$ is a differentiable function. Prove that the gradient vector at the point $(x, y)$ points in the direction of the greatest rate of change for the function at the point $(x, y)$, and the magnitude of the gradient vector is that rate of change. Hint: Use the formula for the directional derivative for a differentiable function, and then use the geometric definition of the dot product.

9. Let $f(x, y) = x^2 + \ln(y)$.
   (a) Find the average rate of change of $f$ as you go from $(3, 1)$ to $(1, 2)$.
   (b) Find the instantaneous rate of change of $f$ as you leave the point $(3, 1)$ heading toward $(2, 1)$.

10. Let $f(x, y)$ be differentiable at the point $(2, 1)$. The directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction towards the point $(1, 3)$ is $\frac{-2}{\sqrt{5}}$, and the directional derivative of $f(x, y)$ at the point $(2, 1)$ in the direction towards the point $(5, 5)$ is 1. Compute $f_x(2, 1)$ and $f_y(2, 1)$.

11. Consider the function,

$$f(x, y) = \begin{cases} x, & \text{if } y = 0 \\ 0, & \text{if } y \neq 0 \end{cases}$$

   (a) Find $f_{\vec{u}}(0, 0)$ where $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$.

   (b) Find $f_x(0, 0)\frac{1}{\sqrt{2}} + f_y(0, 0)\frac{1}{\sqrt{2}}$. What do you notice?

12. Find the equation of the tangent plane at $(2, 3, 1)$ to the surface $x^2 + y^2 - xyz = 7$. Do this in two ways:

   (a) Viewing the surface as the level surface of a function of three variables, $F(x, y, z)$.

   (b) Viewing the surface as the graph of a function of two variables, $z = f(x, y)$.

13. At the point $(1, -1, 1)$ on the surface $f(x, y) = 4 - x^2 - 2y^2$, calculate a vector perpendicular to the surface and a vector tangent to the curve of steepest ascent.

14. Suppose that $w = f(x(u, v), y(u, v))$. Also, suppose you know the following data:

   - $\dfrac{\partial w}{\partial u} = 8$
   - $\dfrac{\partial w}{\partial v} = 4$
   - $\dfrac{\partial x}{\partial u} = 5$
   - $\dfrac{\partial y}{\partial u} = 6$
   - $\dfrac{\partial x}{\partial v} = 1$
   - $\dfrac{\partial y}{\partial v} = 2$

   Use this data to compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

15. Let $z = f(t)g(t)$. Use the chain rule applied to $h(x, y) = f(x)g(y)$ to show that $\frac{dz}{dt} = f'(t)g(t) + f(t)g'(t)$. In other words, you will show that the one variable product rule for differentiation is a special case of the two variable chain rule.

16. Consider the following function.

$$f(x, y) = \begin{cases} \frac{xy^3 - x^3y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

   (a) Use the limit definition of the partial derivative to compute $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Find formulas for $f_x(x,y)$ and $f_y(x,y)$ when $(x,y) \neq (0,0)$.

(c) Use part (a) and part (b), in addition to the limit definition of the partial derivative function, to compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$. What do you notice?

17. Determine if the following function is differentiable at $(0,0)$.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

18. Find $A$ and $B$ so that $f(x,y) = x^2 + Ax + y^2 + B$ has a local minimum value of 20 at $(1,0)$.

19. Draw a possible contour diagram of $f$ such that $f_x(-1,0) = 0$, $f_y(-1,0) < 0$, $f_x(3,3) > 0$, and $f_y(3,3) > 0$, and $f$ has a local maximum at $(3,-3)$.

20. A closed rectangular box with faces parallel to the coordinate planes has one bottom corner at the origin and the opposite top corner in the first octant on the plane $3x + 2y + z = 1$. What is the maximum volume of such a box?

21. Evaluate the following integrals:

(a) $\displaystyle\int_0^1 \int_y^1 e^{x^2} dxdy$

(b) $\displaystyle\int_0^3 \int_{y^2}^9 y \sin(x^2) dxdy$

22. Sketch the region of integration:

$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)dydxdz$$

23. Write a triple integral, including limits of integration, that gives the specified volume of the region between $z = x + y$, $z = 1 + 2x + 2y$, and above $0 \leq x \leq 1$, $0 \leq y \leq 2$.