MATH 2400: Calculus III, Fall 2013
FINAL EXAM

December 16, 2013

YOUR NAME:

Circle Your Section

001 E. Angel .... (9am)
002 E. Angel .... (10am)
003 A. Nita .... (11am)
004 K. Selker .... (12pm)
005 I. Mishev .... (1pm)
006 C. Farsi .... (2pm)
007 R. Rosenbaum .... (3pm)
008 S. Henry .... (3pm)

Important note: SHOW ALL WORK. BOX YOUR ANSWERS. Calculators are not allowed. No books, notes, etc. Throughout this exam, please provide exact answers where possible. That is: if the answer is 1/2, do not write 0.499 or something of that sort; if the answer is π, do not write 3.14159 or something of that sort.

Problem	Points	Score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
TOTAL	100	

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

SIGNATURE:

1. (10 points) For each of the theorems named below you are to select the appropriate assumptions from the following list and to write out the theorem's conclusion.

List of Assumptions:

(A) W is a solid region whose boundary S is a piecewise smooth surface given the outward orientation, and $\vec{F}$ is a smooth vector field on an open region containing W and S.

(B) C is a piecewise smooth oriented path with starting point P and ending point Q, and f is a function whose gradient is continuous on the path C.

(C) S is a smooth oriented surface with piecewise smooth, oriented boundary C whose orientation is given from the orientation on S and by the right hand rule, and $\vec{F}$ is a smooth vector field on an open region contatining S and C.

(a) **The Divergence Theorem**

Suppose • (A), (B), (C) [Circle One].

Then, __________________________=__________________________.

(b) **Stokes' Theorem**

Suppose • (A), (B), (C) [Circle One].

Then, __________________________=__________________________.

2. (10 points) Compute algebraically the following quantities.

(a) $\operatorname{div}(\operatorname{curl}\vec{F})$, where $\vec{F}$ is a smooth vector field on $\mathbb{R}^3$.

(b) $\operatorname{curl}(\operatorname{grad} f)$, where f is a smooth scalar field on $\mathbb{R}^3$.

3. (20 points) A hurricane is swirling around the z axis. In the region where $\sqrt{x^2+y^2} < 5$ (the 'eye' of the hurricane) the winds are calm. In the region where $5 \leq \sqrt{x^2+y^2} \leq 100$ and $0 \leq z \leq 10$ the wind velocity $\vec{v}$ is given by

$$\vec{v}(x,y,z) = \left(\sqrt{x^2+y^2} - 5\right) \frac{-y\,\vec{i} + x\,\vec{j}}{\sqrt{x^2+y^2}}.$$

(a) Compute $\operatorname{curl}\vec{v}$. (Hint: If $\vec{v} = \phi\vec{F}$, then $\operatorname{curl}\vec{v} = \nabla \times (\phi\vec{F}) = \nabla\phi \times \vec{F} + \phi\,(\nabla \times \vec{F}) = (\operatorname{grad}\phi) \times \vec{F} + \phi\,(\operatorname{curl}\vec{F})$.)

(b) Let C_1 be the circle of radius 20 in the xy plane, centered on the origin and oriented counterclockwise. Calculate $\oint_{C_1} \vec{v} \cdot d\vec{r}$.

(c) Let C_2 be the circle of radius 5 in the xy plane, centered on the origin and oriented counterclockwise. Calculate $\oint_{C_2} \vec{v} \cdot d\vec{r}$.

(d) Let S be the region in the xy plane bounded by C_1 and C_2, oriented upward. Assuming $\operatorname{curl} \vec{v} = \left(2 - \frac{5}{\sqrt{x^2+y^2}}\right)\vec{k}$, compute the flux of $\operatorname{curl} \vec{v}$ through S in the upward direction. (Hint: polar coordinates.)

(e) Are the results of (a), (b), (c), and (d) consistent with one another? Explain.

4. (10 points) Let T be the triangle with vertices at $A = (2, 2, 2)$, $B = (4, 2, 1)$, and $C = (2, 3, 1)$. Let $\vec{N} = 2\,\vec{i} + 4\,\vec{j} + 4\,\vec{k}$.

(a) Show that $\vec{N}$ and $\overrightarrow{AB} \times \overrightarrow{AC}$ are parallel to one another.

(b) Find an equation of the form $ax + by + cz = d$ for the plane that T lies in.

(c) Find the area of T.

5. (10 points) Determine for each of the following functions whether it is continuous at the given point. In each case, justify your conclusion.

(a) The function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto 3x + 4y$, at the point $(1, 2)$. Use the ϵ-δ definition of continuity. (Hints: $|a + b| \le |a| + |b|$, $|c| \le \sqrt{c^2 + d^2}$, and $|d| \le \sqrt{c^2 + d^2}$.)

(b) The function

$$g\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto g(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

at the point $(0, 0)$.

(c) The function

$$h\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto h(x,y) = \begin{cases} \frac{\sin((x^2+y^2)^{3/2})}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

at the point $(0,0)$.

6. (10 points) Find and classify the critical points of $f(x, y) = x^3 + y^3 - 3y^2 - 3x + 10$.

7. (10 points)

(a) Use a triple integral to determine the volume of the solid that lies below the surface $z = x + y^2$ and above the square $\{(x, y) \mid 0 \le x \le 3 \text{ and } 0 \le y \le 2\}$.

(b) Find the value of the integral $\displaystyle\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$.

8. (10 points) Use the change of variables $s = x + y, t = y$ to compute the area of the elliptical region bounded by $x^2 + 2xy + 2y^2 = 1$. Show ALL your work, and don't write anything you can't justify.

9. (10 points) Consider the vector field

$$\vec{F}(x,y) = [\sin(y-x)]\,\vec{i} + [e^y(\sin y + \cos y) - \sin(y-x)]\,\vec{j}.$$

(a) Compute the work done by $\vec{F}$ in moving a particle along the line segment from $(0,0)$ to $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$, by setting up and evaluating the line integral directly (not by using the Fundamental Theorem of Calculus for Line Integrals).

(b) Use the Curl Test for Vector Fields in 2-Space to show that $\vec{F}$ is path-independent.

(c) Find a potential function f for $\vec{F}$, and explain how you found it.

$$f(x, y) = \underline{\hspace{6cm}}.$$

(d) Use the Fundamental Theorem of Calculus for Line Integrals to compute the work done by $\vec{F}$ along the line segment given in part (a).