MATH 2400, Analytic Geometry and Calculus 3

LIST OF IMPORTANT DEFINITIONS AND THEOREMS

Use this as you would a dictionary. When you look up a definition and find in it a word whose definition you don't know or don't remember, look back for the definition of that word. Ultimately you will come to words not defined here. For those you have to rely on your intuitive sense of what they plausibly might mean. To fully understand the definitions, study carefully the examples that follow them.

1 Foundations

Definition 1.1. By a **function** is meant a collection of one or more ordered pairs no two of which have the same first term. If f is a function, then

(i) the set of all first terms of ordered pairs in f is called the **domain** of f and is denoted by $\operatorname{dom} f$,

(ii) the set of all second terms of ordered pairs in f is called the **range** of f and is denoted by $\operatorname{ran} f$,

(iii) if x is an element of the domain of the function f, then the second term of the ordered pair in f whose first term is x is called the **value of** f at x and is denoted by $f(x)$ (read f of x); thus the ordered pair is $\langle x, f(x)\rangle$.

Synonyms: A synonym for 'function' is '**mapping**'. When f is referred to as a mapping, then the value of f at x is called 'the **image** of x under (the mapping) f'.

Notation. A standard way to indicate symbolically that f is a mapping (function) whose domain is X and whose range is a subset of Y, and that $f(x)$ is the image of x under f is

$$f\colon X \to Y,\ x \mapsto f(x)\ .$$

Definition 1.2. Because $\operatorname{ran} f \subset Y$, f is said to map X **into** Y, emphasized by the notation $f\colon X \overset{\text{into}}{\to} Y$; if specifically $\operatorname{ran} f = Y$, then f is said to map X **onto** Y, made specific by the notation $f\colon X \overset{\text{onto}}{\to} Y$.

Examples 1.1. Here are some examples of functions:

(1) the identity function on a set X, $\mathrm{id}_X\colon X \to X$, $x \mapsto x$;

(2) the (principle) square-root function, $\sqrt{}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $x \mapsto \sqrt{x}$;

(3) the natural logarithm function, $\ln\colon \mathbb{R}_{>0} \to \mathbb{R}$, $x \mapsto \int_1^x \frac{1}{t}\,dt$.

Definition 1.3. If no two ordered pairs in the function f have the same second term, then f is said to be **invertible**. If f is invertible, then

$$\{\langle y, x\rangle \mid \langle x, y\rangle \text{ is in } f\}$$

is also a function, called the **inverse** (function) of f and denoted by f^{-1}.

Examples 1.2. The functions of Examples 1.1 are all invertible. Their inverses are described by

(1) $\mathrm{id}_X^{-1} = \mathrm{id}_X$;

(2) $\sqrt{}^{-1}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ x \mapsto x^2$;

(3) $\ln^{-1} =$ the exponential function, $\exp\colon \mathbb{R} \to \mathbb{R}_{>0},\ y \mapsto x$ such that $\int_1^x \frac{1}{t}\,dt = y$.

Definition 1.4. If $f\colon X \to Y$, $A \subset X$, and $B \subset Y$, then

1. $f(A) = \{f(x) \mid x \in A\}$, called the **image of** A under f, and
2. $f^{-1}(B) = \{x \mid f(x) \in B\}$, called the **preimage of** B under f. (If f is not invertible, this is an abuse of notation established by convention long ago.)

Definition 1.5. If $f\colon U \to Y$ and $g\colon V \to Z$, then g **is composable with** f means that $f(U) \cap V \neq \emptyset$ (in other words, the range of f and the domain of g have at least one element in common). If g is composable with f, then $\{\langle x, z\rangle \mid f(x) \in \operatorname{dom} g \text{ and } z = g(f(x))\}$ is a function called the **composition of** g **with** f and denoted by $g \circ f$ and by $g(f)$. Thus $g \circ f\colon f^{-1}(\operatorname{ran} f \cap \operatorname{dom} g) \to Z, x \mapsto g(f(x))$.

Definition 1.6. $\mathbb{R} = \{$all real numbers$\}$, $\mathbb{R}^1 = \{$all real number 1-tuples$\}$, $\mathbb{R}^2 = \{$all real number 2-tuples$\}$, $\mathbb{R}^3 = \{$all real number 3-tuples$\}$, and so on. Thus, typical elements of $\mathbb{R}$, $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are x, $\langle x\rangle$, $\langle x, y\rangle$, and $\langle x, y, z\rangle$, where each of x, y, and z is a real number. $\mathbb{R}^1$ is often identified with $\mathbb{R}$ by identifying $\langle x\rangle$ with x.

Notation. $f(\langle x\rangle)$, $f(\langle x, y\rangle)$, and $f(\langle x, y, z\rangle)$ are usually abbreviated to $f(x)$, $f(x, y)$, and $f(x, y, x)$.

Definition 1.7. Functions of the form

$$
\begin{aligned}
&f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto c + mx,\\
&f\colon \mathbb{R}^1 \to \mathbb{R},\ \langle x\rangle \mapsto c + mx,\\
&f\colon \mathbb{R}^2 \to \mathbb{R},\ \langle x, y\rangle \mapsto c + mx + ny,\\
&f\colon \mathbb{R}^3 \to \mathbb{R},\ \langle x, y, z\rangle \mapsto c + mx + ny + lz,
\end{aligned}
$$

where $c, m, n, l \in \mathbb{R}$, are said to be **linear**. If $c = 0$, the linear function f is said to be **homogeneous**.

Definition 1.8. If $f\colon X \to \mathbb{R}$, then by the **graph of** f is meant

- $\{\langle x, f(x)\rangle \mid x \in X\}$ if $X \subset \mathbb{R}$ or $X \subset \mathbb{R}^1$,
- $\{\langle x, y, f(x, y)\rangle \mid \langle x, y\rangle \in X\}$ if $X \subset \mathbb{R}^2$,
- $\{\langle x, y, z, f(x, y, z)\rangle \mid \langle x, y, z\rangle \in X\}$ if $X \subset \mathbb{R}^3$,

Definition 1.9. If $E\colon U \to \mathbb{R}$ and $F\colon V \to \mathbb{R}$, then

- $E(x) = F(x)$ is called **an equation in** x if $U, V \subset \mathbb{R}$ or $U, V \subset \mathbb{R}^1$, and $\{x \mid E(x) = F(x) \text{ is true}\}$ is called its **graph**;
- $E(x, y) = F(x, y)$ is called **an equation in** x **and** y if $U, V \subset \mathbb{R}^2$, and $\{\langle x, y\rangle \mid E(x, y) = F(x, y) \text{ is true}\}$ is called its **graph**;
- $E(x, y, z) = F(x, y, z)$ is called **an equation in** x**,** y**, and** z if $U, V \subset \mathbb{R}^3$, and $\{\langle x, y, z\rangle \mid E(x, y, z) = F(x, y, z) \text{ is true}\}$ is called its **graph**;
- $E(x, y, z, w) = F(x, y, z, w)$ is called **an equation in** x**,** y**,** z**, and** w if $U, V \subset \mathbb{R}^4$, and $\{\langle x, y, z, w\rangle \mid E(x, y, z, w) = F(x, y, z, w) \text{ is true}\}$ is called its **graph**.

Theorem 1.1. If $E\colon U \to \mathbb{R}$, $F\colon V \to \mathbb{R}$, and $f\colon X \to \mathbb{R}$, then

- the graph of $E(x, y) = F(x, y)$ is the graph of f if $U, V \subset \mathbb{R}^2$, $X \subset \mathbb{R}$ or $X \subset \mathbb{R}^1$, $E(x, y) = y$, and $F(x, y) = f(x)$;
- the graph of $E(x, y, z) = F(x, y, z)$ is the graph of f if $U, V \subset \mathbb{R}^3$, $X \subset \mathbb{R}^2$, $E(x, y, z) = z$, and $F(x, y, z) = f(x, y)$;
- the graph of $E(x, y, z, w) = F(x, y, z, w)$ is the graph of f if $U, V \subset \mathbb{R}^4$, $X \subset \mathbb{R}^3$, $E(x, y, z, w) = w$, and $F(x, y, z, w) = f(x, y, z)$.

Definition 1.10. If each of H and K is a set, then by the **cartesian product of** H **with** K, denoted by $H \times K$, is meant the set of all ordered pairs $\langle h, k\rangle$ such that $h \in H$ and $k \in K$, which is to say that $H \times K = \{\langle h, k\rangle \mid h \in H \text{ and } k \in K\}$.

Definition 1.11. If each of H, K, and L is a set, then by the **cartesian product of** H **with** K **with** L, denoted by $H \times K \times L$, is meant the set of all ordered triples $\langle h, k, l\rangle$ such that $h \in H$, $k \in K$, and $l \in L$, which is to say that $H \times K \times L = \{\langle h, k, l\rangle \mid h \in H,\ k \in K, \text{ and } l \in L\}$.

Examples 1.3. Here are some examples of cartesian products:

(1) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$;

(2) $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$;

(3) $[a, b] \times [c, d] = \{\langle x, y\rangle \mid a \le x \le b \text{ and } c \le y \le d\}$ (a closed rectangular region in $\mathbb{R}^2$);

(4) $(a, b) \times (c, d) = \{\langle x, y\rangle \mid a < x < b \text{ and } c < y < d\}$ (an open rectangular region in $\mathbb{R}^2$);

(5) $[a, b] \times (c, d) = \{\langle x, y\rangle \mid a \le x \le b \text{ and } c < y < d\}$ (a rectangular region in $\mathbb{R}^2$ that is not open and not closed);

(6) $[a, b) \times (c, d] = \{\langle x, y\rangle \mid a \leq x < b \text{ and } c < y \leq d\}$ (a rectangular region in $\mathbb{R}^2$ that is not open and not closed);

(7) $[a, b] \times [c, d] \times [e, f] = \{\langle x, y, z\rangle \mid a \leq x \leq b,\ c \leq y \leq d, \text{ and } e \leq z \leq f\}$ (a closed rectangular region in $\mathbb{R}^3$);

(8) $(a, b) \times (c, d) \times (e, f) = \{\langle x, y, z\rangle \mid a < x < b,\ c < y < d, \text{ and } e < z < f\}$ (an open rectangular region in $\mathbb{R}^3$).

(9) $(a, b] \times (c, d) \times (e, f] = \{\langle x, y, z\rangle \mid a < x \leq b,\ c < y < d, \text{ and } e < z \leq f\}$ (a rectangular region in $\mathbb{R}^3$ that is not open and not closed).

2 Continuous Functions and Limits

Definition 2.1. By a **metric space** is meant a set S equipped with a **metric** (also called a **distance function**) $d\colon S \times S \to \mathbb{R}_{\geq 0}, \langle p, q\rangle \mapsto d(p, q)$ (short for $d(\langle p, q\rangle)$) such that if each of p, q, and r is in S, then

(i) $d(p, p) = 0$,

(ii) $d(p, q) > 0$ if $p \neq q$,

(iii) $d(p, q) = d(q, p)$, and

(iv) $d(p, r) \leq d(p, q) + d(q, r)$.

Examples 2.1. Here are some examples of metric spaces:

(1) $S = \mathbb{R}$, $d(x, a) = |x - a|$;

(2) $S = \mathbb{R}^1$, $d(\langle x\rangle, \langle a\rangle) = |x - a|$;

(3) $S = \mathbb{R}^2$, $d(\langle x, y\rangle, \langle a, b\rangle) = \sqrt{(x-a)^2 + (y-b)^2}$;

(4) $S = \mathbb{R}^3$, $d(\langle x, y, z\rangle, \langle a, b, c\rangle) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

Convention. Whenever $\mathbb{R}$, $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$ is used as a metric space, its metric is understood to be that of the preceding example, unless some other metric is specified explicitly.

Definition 2.2. If S is a metric space with metric d, $\hat{S}$ is a metric space with metric $\hat{d}$, and $f\colon S \to \hat{S}$, then f is **continuous at** p means that

(i) $p \in S$, and

(ii) if ϵ is a positive number, then there is a positive number δ such that if q is in S and $d(q, p) < \delta$, then $\hat{d}(f(q), f(p)) < \epsilon$.

That f is **discontinuous at** p means that $p \in S$ and f is not continuous at p. If $T \subset S$, then f is **continuous on** T means that f is continuous at every element p of T. That f is **continuous** means that f is continuous on S (in other words, is continuous at every element p of its domain).

Examples 2.2. Here are some examples of continuous functions:

(1) $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto 3x$, is continuous at the number a because, with $S = \mathbb{R}$ and $\hat{S} = \mathbb{R}$,

$$\hat{d}(f(x), f(a)) = |f(x) - f(a)| = |3x - 3a| = 3|x - a| = 3\, d(x, a) < \epsilon$$

if $d(x, a) < \delta$, where $\delta = \epsilon/3$ (or any smaller positive number). f is continuous because it is continuous at every number a in its domain $\mathbb{R}$.

(2) $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto 7x^2$, is continuous at the number a because if $|x-a| < b$ (any positive number), then $|x| = |(x-a)+a| \le |x-a| + |a| < b + |a|$, so

$$|f(x) - f(a)| = 7|x^2 - a^2| = 7|x+a||x-a| \le 7(|x|+|a|)|x-a| < 7(b+2|a|)|x-a| < \epsilon$$

if $|x-a| < \bar{\delta}$, where $\bar{\delta} = \epsilon/7(b+2|a|)$, and therefore, with $S = \mathbb{R}$ and $\hat{S} = \mathbb{R}$, $\hat{d}(f(x), f(a)) < \epsilon$ if $d(x,a) < \delta$, where $\delta = \min\{b, \bar{\delta}\}$ (or any smaller positive number). f is continuous because it is continuous at every number a in its domain $\mathbb{R}$.

(3) $f\colon \mathbb{R}^2 \to \mathbb{R}, \langle x, y\rangle \mapsto xy$, is continuous at $\langle a, b\rangle$ because, with $S = \mathbb{R}^2$ and $\hat{S} = \mathbb{R}$,

$$\begin{aligned}
\hat{d}(f(x,y), f(a,b)) &= |xy - ab| \\
&= |(x-a)b + a(y-b) + (x-a)(y-b)| \\
&\le |x-a||b| + |a||y-b| + |x-a||y-b| \\
&= \sqrt{(x-a)^2}\,|b| + |a|\sqrt{(y-b)^2} + \sqrt{(x-a)^2}\sqrt{(y-b)^2} \\
&\le \sqrt{(x-a)^2 + (y-b)^2}\,|b| + |a|\sqrt{(x-a)^2 + (y-b)^2} \\
&\qquad + \sqrt{(x-a)^2 + (y-b)^2}\sqrt{(x-a)^2 + (y-b)^2} \\
&= \left(|a| + |b| + \sqrt{(x-a)^2 + (y-b)^2}\right)\sqrt{(x-a)^2 + (y-b)^2} \\
&= \big(|a| + |b| + d(\langle x, y\rangle, \langle a, b\rangle)\big)\, d(\langle x, y\rangle, \langle a, b\rangle) \\
&< \big(|a| + |b| + c\big)\, d(\langle x, y\rangle, \langle a, b\rangle) \\
&\qquad \text{(if } c \text{ is any positive number and } d(\langle x, y\rangle, \langle a, b\rangle) < c) \\
&< \epsilon
\end{aligned}$$

if $d(\langle x, y\rangle, \langle a, b\rangle) < \bar{\delta}$, where $\bar{\delta} = \epsilon/(|a| + |b| + c)$, and therefore $\hat{d}(f(x,y), f(a,b)) < \epsilon$ if $d(\langle x, y\rangle, \langle a, b\rangle) < \delta$, where $\delta = \min\{c, \bar{\delta}\}$. f is continuous because it is continuous at every element $\langle a, b\rangle$ in its domain $\mathbb{R}$.

Theorem 2.1. If S is a metric space with metric d, and $S^\star$ is a subset of S, then $S^\star$ is a metric space with metric $d^\star\colon S^\star \times S^\star \to \mathbb{R}_{\ge 0}, \langle p, q\rangle \mapsto d(\langle p, q\rangle)$, which is to say that $d^\star(\langle p, q\rangle) = d(\langle p, q\rangle)$ if $\langle p, q\rangle \in S^\star$.

Convention. The elements of metric spaces are often referred to as 'points', even if they are not points in the geometric sense.

Definition 2.3. If S is a metric space with metric d, and $S^\star$ is a subset of S, then by a **limit point** of $S^\star$ is meant a point p of S such that if δ is a positive number, then there is in $S^\star$ a point q *distinct* from p such that $d(q, p) < \delta$.

Definition 2.4. If S is a metric space with metric d, $\hat{S}$ is a metric space with metric $\hat{d}$, $S^\star$ is a subset of S, and $f\colon S^\star \to \hat{S}$, then $f(q)$ **approaches** L **as** q **approaches** p means that

(i) p is a limit point of $S^\star$,

(ii) L is a point of $\hat{S}$, and

(iii) if ϵ is a positive number, then there is a positive number δ such that if q is a point of $S^\star$ distinct from p, and $d(q,p) < \delta$, then $\hat{d}(f(q), L) < \epsilon$.

Theorem 2.2. If $f(q)$ approaches L as q approaches p, and $f(q)$ approaches L' as q approaches p, then $L' = L$.

Definition 2.5. If $f(q)$ approaches L as q approaches p, then L is called the **limit of** f **at** p and is denoted by $\lim_{q\to p} f(q)$.

Theorem 2.3. If $S^\star$ is a subset of the metric space S, $\hat{S}$ is a metric space, $f\colon S^\star \to \hat{S}$, and p is a point of $S^\star$ that is also a limit point of $S^\star$, then f is continuous at p if and only if $f(q)$ approaches $f(p)$ as q approaches p, which is to say that $f(p) = \lim_{q\to p} f(q)$.

Examples 2.3. In each of Examples 2.2 the point p ($= a$ in (1) and (2), $= \langle a, b\rangle$ in (3)) is a limit point of $S^\star$ ($= S$ in (1), (2), and (3)), and $f(p) = \lim_{q\to p} f(q)$.