Silverman Conference - Brown University - 11/15 August 2015

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1. Silverman's Bounded Height Theorem

I start my talk by recalling (a version of) Silverman's Bounded Height Theorem. I would like to point out immediately that I shall stick to a very special case of his results in this direction: this is both for clarity and because this shall better fit in the applications that I intend to mention.

Let $\mathcal{L}$ denote the Legendre elliptic scheme over $\mathbb{P}_1 \setminus \{0, 1, \infty\}$, essentially defined by

$$\mathcal{L}: \qquad y^2 = x(x-1)(x-\lambda), \qquad \lambda \in \mathbb{P}_1 \setminus \{0, 1, \infty\}.$$

If we homogenize the equation with respect to x, y, this represents a surface in $\mathbb{P}_2 \times \mathbb{P}_1 \setminus \{0, 1, \infty\}$ with a map λ to $\mathbb{P}_1 \setminus \{0, 1, \infty\}$, whose fiber above a point c is the elliptic curve $\mathcal{L}_c$ with the written Weierstrass equation after substitution $\lambda \to c$.

We may view this also as an elliptic curve defined over the rational function field field $\mathbb{Q}(\lambda)$.

Suppose now we have points $\sigma_1, \ldots, \sigma_r$ on this curve, with coordinates which are *algebraic functions of* λ. [1]

EXAMPLE: $\sigma = (2, \sqrt{2(2-\lambda)})$, a point with constant abscissa. It is in fact well-defined only on the curve $B : \eta^2 = 2 - \lambda$.

These algebraic functions may be not well-defined on $\mathbb{P}_1$, and for this reason we extend the base $\mathbb{P}_1 \setminus \{0, 1, \infty\}$ to an affine smooth curve B (defined e.g. over $\overline{\mathbb{Q}}$) with a rational map denoted also $\lambda : B \to \mathbb{P}_1 \setminus \{0, 1, \infty\}$.

Then we consider the (fibered) product $\mathcal{L} \times_{\mathbb{P}_1 \setminus \{0,1,\infty\}} B =: \mathcal{L}_B$. Note that this is defined by the same equation as above, the only difference being that λ is a function on B rather than $\mathbb{P}_1$.

This now has a map $\pi : \mathcal{L}_B \to B$ with fibers $\mathcal{L}_b = \pi^{-1}(b)$.

In this larger realm, the points σ_i may be viewed as *sections (of* π*)* $\sigma_i : B \to \mathcal{L}_B$. So each σ_i associates to a point $b \in B$ a point $\sigma_i(b) \in \mathcal{L}_b$. [2]

We further suppose that the sections as well are defined over $\overline{\mathbb{Q}}$.

As mentioned, the σ_i are in fact points in the Mordell-Weil group $\mathcal{L}(\overline{\mathbb{Q}}(B))$ over the function field of B, and we now assume that they are *independent*, i.e. if $m_1, \ldots, m_r \in \mathbb{Z}$ are integers not all zero then

$$m_1\sigma_1 + \ldots + m_r\sigma_r \neq 0,$$

which also means that the sum is not *identically* (or *generically*) zero on B.

It is a natural question, which indeed arises in several contexts, to ask:

Question: *For which points* $b \in B(\overline{\mathbb{Q}})$ *do the values* $\sigma_1(b), \ldots, \sigma_r(b)$ *remain independent on* $\mathcal{L}_b$*?*

- Work of Néron, using version of Hilbert's Irreducibility Theorem (but with additional arguments from the Mordell-Weil theorem) proved that the independence of the $\sigma_i(b)$ holds as soon as $\lambda(b)$ lies in $\mathbb{P}_1(\mathbb{Q})$, but *outside a certain 'thin' set of* $\mathbb{P}_1(\mathbb{Q})$.

We skip here precise definitions and only say that such sets may be proved to be actually thin, i.e. in a sense 'sparse'. However, (i) they may be infinite, and (ii) the result applies at most to points b of *bounded degree* over $\mathbb{Q}$. So, this does not say much for general algebraic points of B.

Now, J. Silverman around 1980 proved results which as a very special case imply e.g.

Theorem 1.1 (Silverman 1981)**.** *The set of* $b \in B(\overline{\mathbb{Q}})$ *for which* $\sigma_1(b), \ldots, \sigma_r(b)$ *are dependent on* $\mathcal{L}_b$ *has bounded height.*

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[1] There are too few points with coordinates in $\mathbb{Q}(\lambda)$: they are $(0, 0), (1, 0), (\lambda, 0)$ and the point at infinity.

[2] Sometimes we shall skip such precisions and speak of *values of* σ *at* $\lambda = c \in \mathbb{C}$.

I take it for granted the concept of height here; suffices it to say that a well-known (easy) theorem of Northcott implies that `there are only finitely many points of bounded degree and bounded height.`

So in particular this substantially strengthens Néron's result mentioned before.

An interesting statement is obtained already for $r = 1$, and we shall soon mention applications in joint work with Masser; we have

Corollary 1.2. *If $\sigma : B \in \mathcal{L}_B$ is a non-torsion section, then for $b \in B(\overline{\mathbb{Q}})$, $\sigma(b)$ may be torsion at most for a set of points of bounded height.*

EXAMPLE: The point $(2, \sqrt{2(2-c)})$ on the curve $y^2 = x(x-1)(x-c)$ may be torsion only for a set of algebraic numbers c of bounded height. (This set is provably infinite.)

It is now a few years since Masser asked what kind of supplementary conclusion we may draw on prescribing simultaneously that *two* sections are torsion. In a series of papers we succeeded to obtain a number of finiteness results in this direction, of which a rather special case is the following

Theorem 1.3 (Masser, Z. $\approx$ 2010)**.** *Let $u \neq v$ be distinct complex numbers outside $\{0, 1\}$. Then there are only finitely many $c \in \mathbb{C}$ for which $(u, \sqrt{u(u-1)(u-c)})$ and $(v, \sqrt{v(v-1)(v-c)})$ are both torsion on $\mathcal{L}_c$.*

Silverman's theorem is crucial in more than one aspect of our proof of this theorem.

This is the starting point of further joint work with Masser, in which we considered general sections to

- a square of an elliptic pencil, e.g. $\mathcal{L}_\lambda \times \mathcal{L}_\lambda$,
- a produce of non-isogenous elliptic pencils, e.g. $\mathcal{L}_\lambda \times \mathcal{L}_{2\lambda}$,
- a pencil of simple abelian surfaces.

These cases presented analogies but also different aspects. They are cases of the so-called *Pink's conjectures.*

Remark 1.4. `Other results.` It is important to remark that for brevity and simplicity I have stated only very special corollaries of other highly significant results by Silverman. Indeed, he worked (also with Tate) with general families of abelian varieties, over an arbitrary base B, comparing 'functional (canonical) heights' with 'specialized (canonical) heights'.

His results consider also the 'constant part' and take stronger form when the Néron-Severi group of B is cyclic, which is e.g. the case of curves.

- Manin and Demianenko had proved results in a somewhat similar spirit, but as far as I know only for constant abelian varieties and points defined over a number field (again under the above assumption on NS and a slightly different independence assumption - see also below).

- Silverman's Theorem has inspired a wealth of research, of which I shall succeed to mention only a very small part.

Also, beyond the toric analogues which I shall soon discuss, there are analogous height bounds in the context of algebraic dynamics, some of which have been obtained by Silverman himself (also jointly with other authors, e.g. Call and Kawaguchy).

Again in the context of dynamics, the above theorem has been proved with other methods by DeMarco, Wang, Ye, whereas a dynamical analogue for the family $x^2 + \lambda$ was dealt with by M. Baker, DeMarco, with subsequent extensions by Tucker, Ghioca, Hsia.

And P. Habegger proved a rather delicate bounded-height result when B has arbitrary dimension and there are at least $\dim B$ sections, with a bound holding on a certain open subset of B. He used this to derive a finiteness result like the above one, but for *three points defined on a surface.*

2. Algebraic tori

The said results by Silverman concerned abelian schemes, where applications were more numerous. But one could naturally think of other (commutative) group-varieties, the simplest case being that of algebraic tori $\mathbb{G}_m^n$.

Here two important differences are:

(a) `There are no families`: on the one hand, this is analogous to 'isotrivial' families in the abelian case, which represent an obstruction for Silverman's BH theorem (though not for other of his results); on the other hand, this leads to simplifications with respect to a.v.

(b) `The height behaves in a somewhat worse way`: indeed the height is a quadratic form on a.v., but is less regular on tori $\quad\to\quad$ leads to difficulties.

As to (a), we still have the liberty provided by the sections, i.e. we can consider the 'constant' family

$$\mathcal{T} = \mathbb{G}_m^n \times B, \qquad \pi = \pi_2 : \mathcal{T} \to B,$$

(where we stick again to curves B for the moment) and sections $\sigma_1, \ldots, \sigma_r : B \to \mathcal{T}$, assumed to be (generically) independent, i.e.

$$\text{For integers } a_1, \ldots, a_r \text{ not all zero, we have} \qquad \sigma_1^{a_1} \cdots \sigma_r^{a_r} \neq 1.$$

Remark 2.1. We remark that here we could require the stronger `independence modulo nonzero constants`, sometimes a very important difference. Such stronger assumption appears in the quoted work by Manin-Demianenko.

Already the case $n = 1$ is not obvious. We can view $\sigma := (\sigma_1, \ldots, \sigma_r)$ as a single section $\sigma : B \to \mathbb{G}_m^r \times B$, and then the first projection yields a map $\sigma^* := \pi_1 \circ \sigma : B \to \mathbb{G}_m^r$ and a subvariety $X = \sigma^*(B) \subset \mathbb{G}_m^r$ (at most a curve for B a curve).

`Dependence and algebraic subgroups`: A dependence at $b \in B$ may be seen as an intersection point $x = \sigma^*(b) \in X \cap H$, where H is a proper algebraic subgroup of $\mathbb{G}_m^r$, defined by $x_1^{a_1} \cdots x_r^{a_r} = 1$.

The generic-independence assumption says that X *is not contained in any such* H and the stronger independence mod constants means that X *is not contained in any translate* gH.

There are in general infinitely many *dependent* points (i.e. in $\bigcup_{H\ proper} X \cap H$). After sparseness results by Masser, in joint work with Bombieri and Masser we proved in 1999 that

Theorem 2.2 (Bombieri-Masser-Z. 1999)**.** *If $\sigma_1, \ldots, \sigma_r$ are independent <u>modulo constants</u> then $h(x)$ is bounded for $x \in X \cap \bigcup_{\dim H \le r-1} H$.*

Here the union is taken over all proper alg. subgroups. Note that $h(x) \ll 1$ implies $h(b) \ll 1$ if $x = \sigma_1(b)$.

The strong-independence assumption (i.e. mod. constants) may be seen to be necessary.

This result was important in deducing a finiteness when we have *double dependence*, i.e. we intersect with algebraic subgroups of *codimension* 2:

Theorem 2.3 (Bombieri-Masser-Z. 1999)**.** *If $\sigma_1, \ldots, \sigma_r$ are independent modulo constants then the set $X \cap \bigcup_{\dim H \le r-2} H$ is finite.*

`Improvements`: I shall not pursue on this much further, but only say that in the last theorem *the assumption may be relaxed to mere independence*; this was first proved by Maurin (after special cases by BMZ), and is the case of curves of the so-called *Zilber's conjecture* (2002).

`Elliptic case`: An extension to this theorem to an elliptic context, containing as a special case Thm. 1.3, has been recently obtained by Barroero and Capuano.

Question: What happens for $\dim B > 1$?

Again, we may consider the variety X, and $\dim X \le \dim B$.

Now to obtain Bounded Height we must at least impose as many dependencies as $\dim B$. It turned out in examples worked out with Bombieri and Masser that this condition together with (strong) independence is not yet sufficient.

We formulated a *Bounded Height Conjecture* asserting that BH should hold on a certain subset $X_0 \subset X$, which we proved to be Zariski-open (but sometimes empty, EXAMPLES in $\mathbb{G}_m^4$: products of two lines in $\mathbb{G}_m^2$, BH does NOT hold).

We omit definitions and only say that the conjecture was later proved by Habegger, and led to a second proof of Maurin's Theorem in joint work with Bombieri, Habegger and Masser.

3. Pell's equations in polynomials

In the abelian case again, Silverman's theorem was most crucial in the proofs by Masser and myself of special cases of the so-called *Pink's conjectures*, which may be seen as a kind of Manin-Mumford for abelian families. An instance is the *two points*-theorem mentioned before.

Also, rather concretely, this context is related to a polynomial variant of Pell's equation, studied already by Abel (1826), and I shall use this connection to illustrate.

For instance, consider the (Pell) equation

$$x^2 - (t^4 + t + 1)y^2 = 1,$$

to be solved in complex polynomials $x(t), y(t) \neq 0$; this polynomial case is different from the classical one with integers, where the simplest necessary conditions for solvability are also sufficient; indeed, it can be shown that this equation is not solvable. But it is in the case

$$x^2 - (t^4 + t)y^2 = 1, \quad \text{where we can take} \qquad x(t) = 2t^3 + 1, y(t) = 2t.$$

Question: *For which* $c \in \mathbb{C}$ *is* $\quad x^2 - (t^4 + t + c)y^2 = 1 \quad$ *likewise solvable ?*

We say that the relevant polynomial is *Pellian* in that case.

Well, it is not too difficult to show this cannot be done identically, but it can be done for infinitely many cs, actually all being algebraic. Examples: $c = 0, 1/2, \zeta_3/2, ...$). Less clear is that these numbers *have bounded height*; actually, this follows again from Silverman's theorem.

Let us briefly explain the link: associated to the Pell equation is the family of (genus 1) curves

$$H_\lambda : u^2 = t^4 + t + \lambda.$$

There are two points at infinity (in a smooth model), the poles of t, denoted ∞_+, ∞_-. Their difference

$$\delta = \delta_\lambda := \infty_+ - \infty_-$$

is a divisor of degree 0, and thus defines a point (denoted again δ_λ) in the Jacobian J_λ of H_λ. So we have a section δ of a family of Jacobians.

Now, it is a well-known fact since long ago that:

Criterion: *Pell is solvable for* $\lambda = c \in \mathbb{C}$ *if and only if* δ_c *is torsion on* J_c*.*

Then the above corollary of Silverman's theorem (applied to the present family of Jacobians, which may be shown to satisfy certain necessary assumptions) immediately yields what stated.

Also, like for a previous case, Masser and I obtained some finiteness, for other Pell's equations, this time in higher genus. For instance we have:

Theorem 3.1. *(Masser, Z., 2013) There are only finitely many* $c \in \mathbb{C}$ *such that the Pell's equation* $x^2 - (t^6 + t + c)y^2 = 1$ *is solvable in nonzero polynomials.*

Again, we have a section of a family of Jacobians, and again Silverman's theorem yields the crucial information that the relevant numbers c (necessarily algebraic) have bounded height. This ingredient allows further steps in our proof method to work. (Note that this time the Jacobians have dimension $2 > 1$, and this is the main reason motivating finiteness - we do not pause here on a certain a bit subtler point.)

4. Algebraic coefficients versus transcendental coefficients

- `Transcendental coefficients`: While the height theorems above implicitly require algebraic quantities, the finiteness results may be formulated over $\mathbb{C}$, as was done for instance by Pink for his already mentioned conjectures. For instance, one could ask about the solvability in polynomials of the Pell's equations inside the family

$$y^2 - (t^6 + \pi t^2 + et + \lambda)x^2 = 1, \qquad x, y \in \mathbb{C}[t] \setminus \mathbb{C}.$$

- `Further parameters`: Another possible generalization of the context is obtained by introducing further parameters, in addition to λ, and then asking for the subvarieties of the parameter-space where the Pell's equation is solvable; for the present case of genus 2, the **expectation** is that *there are only finitely many hypersurfaces where this can happen.* [3]

[3] We tacitly assume that certain natural obstructions, which can be classified, are ruled out.

However these two issues are strongly related, because independent transcendentals can be viewed as new variables. (For instance, the displayed equation corresponds to a problem in 2 or 3 dimensions according as π, e are alg. dependent or not.)

`Specializing parameters and problems which arise`: These extensions to several variables might appear innocuous, or even easier, as often happens when we go from number fields to function fields. For instance, one might think of specializing the new variables in order to reduce to the (known) algebraic case. In principle this should be a sensible attempt, especially since the proof pattern that we adopted does not fully work outside $\overline{\mathbb{Q}}$ (for instance, Northcott's theorem is missing, but there are also further obstacles).

However it turned out that troublesome obstacles appear in trying to *prove that specialization does not lead to degenerate cases.* (Together with Bombieri and Masser we had already experimented similar difficulties in trying to extend Maurin's theorem to complex curves, which we eventually did.)

For instance, if we want to exclude infinitely many Pellian polynomials $t^6 + \pi t^2 + t + c_n$, then c_n would be certain algebraic functions of π, and then we could try to specialize π, seen as a variable, e.g. to $\pi = 2$, obtaining Pellian polynomials $t^6 + 2t^2 + t + c_n(2)$.

Now two difficulties arise:

(i) c_n could be undefined at (or, rather, above) 2;

(ii) the values $c_n(2)$, $n = 1, 2, \ldots$, could make up a finite set (and then no contradiction would appear in applying the known algebraic case).

In other words, it could happen that the *torsion curves* which arise **escape towards the boundary** (and fail to have infinite intersection with any prescribed curve defined over $\overline{\mathbb{Q}}$).

Eventually, in recent collaboration with P. Corvaja and Masser, we have overcome both issues; one of the main tools is a *uniform version of Silverman's theorem*, i.e. with a base curve B which varies (also defined over a number field of any degree), taking into account the height of equations which define it. More precisely, here is a special instance:

`Uniform Silverman`: *Let σ be a section of a family of a.v over a surface $S/\overline{\mathbb{Q}}$, covered by a family of curves $B_\theta = \pi^{-1}(\theta)$, fibers of a morphism π of S to a curve. Then, if σ restricted to B_θ is non-torsion, we have $h(b) \leq c(h(\theta) + 1)$ for any point $b \in B_\theta$ such that $\sigma(b)$ is torsion (where c depends only on S, σ, π).*

This uniform version (and more) may be obtained on following closely Silverman's ideas. Then, we apply this result to families of curves of bounded height, defined over cyclotomic fields of growing degree. This forces certain Unlikely Intersections to stay inside a suitable compact set, and this allows other arguments to work. We skip any other detail here.

5. Bounded Height with a weaker condition ?

In the last part of my talk I shall discuss a possible generalization of Silverman's theorem, which is still unknown except for the toric case, which has been worked out in recent joint work with F. Amoroso and Masser.

Let us go back to the Jacobian family

$$J_\lambda = \text{ Jacobian of } H_\lambda : u^2 = t^6 + t + \lambda$$

which we have met in connection with the Pell's equation $x^2 - (t^6 + t + \lambda)y^2 = 1$.

We had a section δ whose value at $\lambda = c$ is defined as the class in J_c of the divisor $\infty_+ - \infty_-$.

We have remarked that solvability of Pell for $\lambda = c$ amounts to δ_c being torsion on J_c, and these cs have bounded height by Silverman's theorem.

`A new condition`. Now, it is of some interest, also from the *Pell viewpoint*, to ask for the c such that `some` multiple $n\delta_c$ is not necessarily 0 but rather, inside a prescribed proper subvariety of J_c, e.g. inside H_c (thought as already embedded in J_c e.g. by $x \mapsto$class of $(x) - (\infty_+)$).

This condition is much weaker than being torsion, but still we do not expect it to be generically satisfied, except possibly for certain fixed integers n. Also, for any given c, it may be satisfied by at most finitely many n (Mordell-Lang, but Chabauty suffices).

Remark 5.1. Note that the condition that a multiple of a point lies in a curve inside its Jacobian is quite familiar and natural in Diophantine Geometry. And one could naturally work also with linear combinations of several sections.

Concerning the Pell aspect, these numbers c are those such that the 'quasi-Pell' equation $x^2 - (t^6 + t + c)y^2 =$ linear polynomial, is solvable.

For this specific example, let us consider only the $n > 3$ (since for $n = 3$ the condition holds identically).

`The (vague) question`: What can be said about the set $\mathcal{E}$ of such c s ?

Inspection suggests that it is not even obvious that $\mathcal{E}$ does not exhaust $\overline{\mathbb{Q}}$, in spite of the fact that such a fact (i.e. $\mathcal{E} = \overline{\mathbb{Q}}$) would seem quite unexpected.

Indeed, V. Flynn has proved (by a variant of Chabauty's method) that the complement $\overline{\mathbb{Q}} \setminus \mathcal{E}$ is infinite (actually a more precise statement).

However his (ingenious and somewhat delicate) method is not guaranteed to work in general.

We wonder about the answer to

Question: Does $\mathcal{E}$ have bounded height ?

Flynn's result does not answer this, which, if true, this would represent a sharpening of Silverman's theorem, in a direction different from those already explored.

`A toric version`. Together with Amoroso and Masser we have worked out a toric analogue. Let me state a concrete polynomial special case of this.

Let $f_1, \ldots, f_r \in \overline{\mathbb{Q}}[\lambda]$, $a_1, \ldots, a_r \in \overline{\mathbb{Q}}^*$.

We consider the hyperplane $H : a_1x_1 + \ldots + a_rx_r = 0$ inside $\mathbb{G}_m^r$ and the section

$$\sigma(\lambda) = (f_1(\lambda), \ldots, f_r(\lambda)).$$

We are interested in the set

$$\mathcal{E}_1 = \{c \in \overline{\mathbb{Q}} : [n]\sigma(c) \in H \text{ for some } n \text{ such that the condition does not hold identically}\}$$

In other words, $\mathcal{E}_1$ consists of all the roots of the equations in λ

$$a_1f_1(\lambda)^n + \ldots + a_rf_r(\lambda)^n = 0, \qquad n = 1, 2, \ldots.$$

however excluding possible degeneracies coming from identities.

Now, it is not too difficult, and a pleasant exercise (not obvious to me), to prove that $\overline{\mathbb{Q}} \setminus \mathcal{E}_1$ is infinite, at least assuming that no two of the polynomials are proportional.

Our result confirms actually much more, namely the sought bounded height:

Theorem 5.2. *(Amoroso, Masser, Z.) On the above assumptions, the set $\mathcal{E}_1$ has bounded height.*

Even the case $f_1 = \lambda, f_2 = 1 - \lambda, f_3 = 1$ seems not obvious (here one has to restrict to $n \geq 2$); it had been dealt with by Beukers in the context of a completely different problem.

We can in fact prove more precise results, and more generally for any number of sections.

Our methods are rather involved, with constructions of auxiliary functions and a combinatorial analysis which we have found intricate. Wronskians play a role, and this reminds of other proofs in Dioph. Appr. and Transcendence.

Indeed, there are several direct applications to finiteness in diophantine problems.

No analogy is in (my) sight which should enable to treat the abelian case; taking into account certain similarities between our proof and proofs of Thue's and related theorems, possibly the version by Bombieri of Vojta's proof of the Mordell conjecture could suggest a good approach.

With this we conclude our talk, leaving these questions for the interested ones in the audience.