Recent Applications of Schmidt's Subspace Theorem

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Abstract. In 2002, Corvaja and Zannier obtained a new proof of Siegel's theorem (on integral points on curves) based on Schmidt's celebrated Subspace Theorem. Soon after that (and based on earlier work), Evertse and Ferretti applied Schmidt's theorem to give diophantine results for homogeneous polynomials of higher degree on a projective variety in $\mathbb{P}^n$. This has led to further work of A. Levin, P. Autissier, M. Ru, G. Heier, and others. In particular, Ru has defined a number, $\operatorname{Nev}(D)$, that concisely describes the best diophantine approximation for an effective divisor D on a projective variety X. In this talk, I will give an overview of this area of research, and indicate how an example of Faltings can be described using $\operatorname{Nev}(D)$.

§1. Introduction

Throughout this talk, k is a number field unless otherwise specified, S_∞ is the set of archimedean places of k, and S is a finite set of places of k.

Definition. For this talk, a variety over a field k is an integral separated scheme of finite type over k, and a curve over k is a variety of dimension 1 over k.

Define norms $\|\cdot\|_v$ on k for all $v \in M_k$ as follows:

$$\|x\|_v = \begin{cases} |\sigma(v)| & \text{if } v \text{ is real, corresponding to } \sigma\colon k \hookrightarrow \mathbb{R}\,; \\ |\sigma(v)|^2 & \text{if } v \text{ is complex, corresponding to } \sigma\colon k \hookrightarrow \mathbb{C}\,; \\ (\mathscr{O}_k : \mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)} & \text{if } v \text{ is finite, corresponding to } \mathfrak{p} \subseteq \mathscr{O}_k \text{, and } x \ne 0\,. \end{cases}$$

(And, of course, $\|0\|_v = 0$ for all v.)

Note that $\|\cdot\|_v$ is not an absolute value if v is complex.

§2. Schmidt's Subspace Theorem

Let $n \in \mathbb{Z}_{>0}$.

For a hyperplane H in $\mathbb{P}^n_k$, define a Weil function $\lambda_H\colon \coprod_v \mathbb{P}^n(\mathbb{C}_v) \to \mathbb{R} \cup \{+\infty\}$ for H by

$$\lambda_{H,v}(P) = -\log \frac{\|a_0x_0 + \dots + a_nx_n\|_v}{(\max \|a_i\|_v)\,(\max \|x_i\|_v)}\,,$$

where $[x_0 : \dots : x_n]$ are homogeneous coordinates for P and H is described by $a_0x_0 + \dots + a_nx_n = 0$.

Then Schmidt's Subspace Theorem is:

Theorem. *Let $\epsilon > 0$, let $C \in \mathbb{R}$, and for each $v \in S$ let $H_{v,0}, \dots, H_{v,n}$ be hyperplanes in $\mathbb{P}^n_k$ in general position. Then there is a finite union Z of proper linear subspaces of $\mathbb{P}^n_k$ such that*

$$\sum_{v\in S} \sum_{i=0}^{n} \lambda_{H_{v,i},v}(P) \le (n+1+\epsilon)h_k(P) + C$$

for all $P \in \mathbb{P}^n(k) \setminus Z$.

More generally:

Theorem. *Let ϵ and C be as above, and let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n_k$ with $\bigcap_{i=1}^q H_i = \emptyset$. Then there is a finite union Z of proper linear subspaces of $\mathbb{P}^n_k$ such that*

$$\sum_{v\in S} \max_{\mathscr{I}} \sum_{i\in\mathscr{I}} \lambda_{H_i,v}(P) \le (n+1+\epsilon)h_k(P) + C$$

for all $P \in \mathbb{P}^n(k) \setminus Z$, where the sum is over all $(n+1)$-element subsets $\mathscr{I}$ of $\{1, \dots, q\}$ such that $\bigcap_{i\in\mathscr{I}} H_i = \emptyset$.

- Roth's theorem is the $n = 1$ case of the above.
- This is a special case of my conjecture.

§3. More General Diophantine Approximation Statements

How about: more general D in $\mathbb{P}^n$, or on projective X?
Basic method (Corvaja and Zannier 2002; Evertse and Ferretti 2003?):

Find an embedding of X or $\mathbb{P}^n$ into $\mathbb{P}^N$, such that each irreducible component of D spans a linear subspace of high codimension (relative to n).

Theorem (Evertse-Ferretti). *Let $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^n_k$ of degrees $d_1, \dots, d_q$, respectively, that meet properly. Then for all $\epsilon > 0$ and all $C \in \mathbb{R}$ there is a proper Zariski-closed subset Z of $\mathbb{P}^n$ such that*

$$\sum_{i=1}^{q} \frac{1}{d_j} \sum_{v \in S} \lambda_{D_j,v}(P) \le (n+1+\epsilon) h_k(P) + C$$

holds for all $P \in \mathbb{P}^n(k) \setminus Z$.

Theorem (Levin). *Let X be a projective variety of dimension $n \ge 2$, and let $D = \sum_{i=1}^{q} D_i$ be a divisor on X, with D_i effective big Cartier divisors on X for all i. Then no set of S-integral k-rational points on $X \setminus D$ is Zariski dense.*

Theorem (Autissier). *Let X be a projective variety over k of dimension $d \ge 2$. Let $D_1, \dots, D_{2d}$ be effective ample divisors on X that meet properly. Then no set of S-integral k-rational points on $X \setminus \sum D_i$ is Zariski dense.*

General methods:

- Large divisors (Levin): D large implies degeneracy of integral points
- $\mathrm{Nev}(D)$ (Ru).

§4. Definition of $\operatorname{Nev}(D)$

Definition (Ru). Let D be an effective Cartier divisor on a projective variety X. If X is normal, then we define $\operatorname{Nev}(D)$ as follows:

$$\operatorname{Nev}(D) = \inf_{(N,V,\mu)} \frac{\dim V}{\mu} ,$$

where the inf is taken over all $N \in \mathbb{Z}_{>0}$, all linear subspaces V of $H^0(X, \mathscr{O}(ND))$ with $\dim V \ge 2$, and all $\mu > 0$ that satisfy the following property. For all $P \in X$ there is a basis $\mathscr{B}$ of V such that

$$(*) \qquad \sum_{s\in\mathscr{B}} \operatorname{ord}_E(s) > \mu \operatorname{ord}_E(ND)$$

for all irreducible components E of $\operatorname{Supp} D$ that pass through P.

If there are no such triples, then $\operatorname{Nev}(D) = +\infty$.

If X is not normal, then define $\operatorname{Nev}(D)$ by pulling back to the normalization of X.

Comments

(1). We can change $>$ to $\ge$ in (*).
(2). We can require μ to be rational.
(3). We can impose (*) on all prime divisors E on X that pass through P.
(4). The condition on the triple can be replaced by,

$$\sum_{s\in\mathscr{B}} (s) - \mu ND \quad \text{is effective near } P .$$

(5). D plays two roles in this definition; we can separate them.
(6). We can define $(\mathscr{B}) = \sum_{s\in\mathscr{B}} (s)$.

Definition. Let X be a complete variety, let D be an effective Cartier divisor on X, and let $\mathscr{L}$ be a line sheaf on X. If X is normal then we define

$$\operatorname{Nev}(\mathscr{L}, D) = \inf_{(N,V,\mu)} \frac{\dim V}{\mu} .$$

Here the inf is taken over all $N \in \mathbb{Z}_{>0}$, all linear subspaces $V \subseteq H^0(X, \mathscr{L}^{\otimes N})$ with $\dim V \ge 2$, and all $\mu > 0$ that satisfy the following property. For all $P \in X$ there is a basis $\mathscr{B}$ of V such that

$$(\mathscr{B}) - \mu ND$$

is effective near P. If there are no triples (N, V, μ), then $\operatorname{Nev}(\mathscr{L}, D)$ is defined to be $+\infty$. For general complete varieties X, $\operatorname{Nev}(\mathscr{L}, D)$ is defined by pulling back to the normalization of X.

If L is a Cartier divisor or Cartier divisor class on X, then $\operatorname{Nev}(L, D)$ is defined to be $\operatorname{Nev}(\mathscr{O}(L), D)$. Also define $\operatorname{Nev}(D) = \operatorname{Nev}(D, D)$.

Then the Nev property is:

Proposition. *Let X be a projective variety, and let D be an ample effective Cartier divisor on X. Let $\epsilon > 0$ and let $C \in \mathbb{R}$. Then there is a proper Zariski-closed subset Z of X such that the inequality*

$$\sum_{v \in S} \lambda_{D,v}(P) \le (\operatorname{Nev}(D) + \epsilon) h_D(P) + C$$

holds for all $P \in X(k) \setminus Z$.

Proof. We may assume that X is normal (by pulling back to the normalization).

Given $\epsilon > 0$, find N, V, and μ such that $\frac{\dim V}{\mu} < \operatorname{Nev}(D) + \epsilon$, and such that for all $P \in X$ there is a basis $\mathscr{B}$ of V such that

(1) $$(\mathscr{B}) - \mu N D \text{ is effective near } P .$$

By quasi-compactness of X, we can get by with finitely many bases, say $\mathscr{B}_1, \dots, \mathscr{B}_\ell$.

Fix any basis for V; it defines a rational map $\psi \colon X \dashrightarrow \mathbb{P}^n$, where $n = \dim V - 1$. Also, each $s \in \bigcup \mathscr{B}_i$ corresponds to a linear form on $\mathbb{P}^n$.

Let Σ be a generic subset of $X(k)$. After removing finitely many points, we may assume that none of the $s \in \bigcup \mathscr{B}_i$ vanishes at any point of Σ. Choose Weil functions $\lambda_{s,v}$ for (s), for all $v \in S$ and all $s \in \bigcup \mathscr{B}_i$.

For each $v \in S$,

$$\max_{i=1,\dots,\ell} \sum_{s \in \mathscr{B}_i} \lambda_{s,v} \ge \mu N \lambda_{D,v} + O(1) ,$$

by (1) and properties of Weil functions. Therefore

$$\sum_{v \in S} \max_{i=1,\dots,\ell} \sum_{s \in \mathscr{B}_i} \lambda_{s,v}(P) \ge \mu N m_S(D,P) + O(1)$$

for all $P \in X(k)$ for which the Weil functions are all defined. But, by Schmidt's Subspace Theorem,

$$\sum_{v \in S} \max_{i=1,\dots,\ell} \sum_{s \in \mathscr{B}_i} \lambda_{s,v}(P) \le (\dim V + \epsilon) h_\psi(P) + O(1) .$$

Since $h_\psi(P) \le h_{ND}(P) + O(1)$, the above two inequalities imply

$$\mu N m_S(D,P) \le (\dim V + \epsilon) h_{ND}(P) + O(1) ;$$

therefore

$$m_S(D,P) \le (\operatorname{Nev}(D) + \epsilon') h_D(P) + O(1) . \qquad \square$$

Examples.

(1). If $H_1, \dots, H_q$ are hyperplanes in $\mathbb{P}^n_k$ in general position, and if $D = \sum H_i$, then

$$\mathrm{Nev}(D) \le \frac{n+1}{q} \; .$$

(2). If D is an irreducible hypersurface of degree d in $\mathbb{P}^n_k$, then

$$\begin{aligned}
\mathrm{Nev}(D) &\le \lim_{N\to\infty} \frac{h^0(\mathbb{P}^2, \mathscr{O}(dN))}{\frac{1}{N}\sum_{j=1}^{\infty} h^0(\mathbb{P}^2, \mathscr{O}(dN - jd))} \\
&= \lim_{N\to\infty} \frac{\frac{(dN+1)(dN+2)}{2}}{\frac{1}{N}\sum_{j=1}^{N} h^0(\mathbb{P}^2, \mathscr{O}(N-j)d))} \\
&= \lim_{N\to\infty} \frac{\frac{N}{2} + O(N)}{\frac{N}{6} + O(N)} \\
&= 3 \; .
\end{aligned}$$

Remark. If $\mathrm{Nev}(D) \ge 1$ then the method does not yield anything useful.

Remark. $\mathrm{Nev}(D) < 1$ if and only if ND is essentially very large for some $N > 0$ (Heier-Ru, arXiv).

§5. Faltings' Example

Faltings constructs:

- an irreducible divisor D on $\mathbb{P}^2_k$, smooth except for cusps and ordinary double points;
- a smooth projective surface Y; and
- a finite morphism $\pi\colon Y \to \mathbb{P}^2_k$;

such that

- $K(Y)$ is a Galois extension of $K(\mathbb{P}^2)$ with Galois group $\mathscr{S}_n$ for some n;
- π is etale (or rather, étale) over $\mathbb{P}^2 \setminus D$;
- the divisor π^*D is $2\sum_{1\le i<j\le n} Z_{ij}$, where the Z_{ij} are all Galois conjugate and have smooth supports;
- $Y \setminus \sum Z_{ij}$ as only finitely many integral points, and therefore so does $\mathbb{P}^2 \setminus D$.

Most importantly,

- $Y \setminus \sum Z_{ij}$ does not embed into a semiabelian variety.

Why it works

Zannier; Levin

Let

$$A_i = \sum_{j\ne i} Z_{ij}$$

for all i (define $Z_{ij} = Z_{ji}$ if $i > j$);

$$Z = \sum_{i<j} Z_{ij} \qquad \text{and} \qquad M = \sum A_i = 2Z\ .$$

Proposition. *Let $\alpha \in \mathbb{Q}_{>0}$ and assume that M and $M - \alpha A_i$ are ample for all i. Assume that $n \ge 4$. Let v be a place of k, and fix Weil functions $\lambda_{ij,v}$ for each Z_{ij} at v. Let β be an integer such that $\beta\alpha \in \mathbb{Z}$ and such that βM and all $\beta(M-\alpha A_i)$ are very ample. Fix an embedding $Y \hookrightarrow \mathbb{P}^N_k$ associated to a complete linear system of βM, and regard Y as a subvariety of $\mathbb{P}^N_k$ via this embedding. Then*

(a). *There is a finite list $H_1, \dots, H_q$ of hyperplanes in $\mathbb{P}^N_k$, with associated Weil functions $\lambda_{H_j,v}$ at v for all j, with the following property. Let $\mathscr{J}$ be the collection of all 3-element subsets $J = \{j_0, j_1, j_2\}$ of $\{1, \dots, q\}$ for which $Y \cap H_{j_0} \cap H_{j_1} \cap H_{j_2} = \emptyset$. Then $\mathscr{J} \ne \emptyset$, and the inequality*

$$\max_{J\in\mathscr{J}} \sum_{j\in J} \lambda_{H_j,v}(y) \ge \beta\alpha \sum_{i<j} \lambda_{ij,v}(y) + O(1) \tag{2}$$

holds for all $y \in Y(k)$ not lying on the support of any Z_{ij} or on any of the H_j.

(b). *Let C be an integral curve in Y, not contained in the support of any Z_{ij}. Then there is a finite list $H_1, \dots, H_q$ of hyperplanes, with associated Weil functions as before, with the following property. Let $\mathscr{J}$ be the collection of all 2-element subsets $J = \{j_0, j_1\}$ of $\{1, \dots, q\}$ for which $C \cap H_{j_0} \cap H_{j_1} = \emptyset$. Then $\mathscr{J} \ne \emptyset$, and the inequality*

$$\max_{J\in\mathscr{J}} \sum_{j\in J} \lambda_{H_j,v}(y) \ge \frac{\beta\alpha}{2} \sum_{i<j} \lambda_{ij,v}(y) + O(1) \tag{3}$$

holds for all but finitely many $y \in C(k)$.

In each case the implicit constant in $O(1)$ is independent of y but may depend on all of the other data.

Theorem (Faltings). *If $\alpha > 6$ then no set of S-integral points on $Y \setminus Z$ is Zariski dense.*

Proof. For all S-integral points P outside of some proper Zariski-closed subset,

$$\frac{\alpha}{2} h_{\beta M}(P) = \beta\alpha m_S(Z, P) + O(1) \le \max_{J\in\mathscr{J}} \sum_{j\in J} \lambda_{H_j,v}(P) + O(1) \le (3+\epsilon) h_{\beta M}(P) + O(1) ,$$

a contradiction if the heights go to ∞. □

Where's Nev ?

Definition. Let X be a complete variety, let $\mathscr{L}$ be a line sheaf on X, and let D be an effective $\mathbb{Q}$-Cartier divisor on X. Define

$$\operatorname{Nev}'(\mathscr{L}, D) = \inf_{N,V,\mu} \frac{\dim V}{\mu} ,$$

where the inf passes over all (N, V, μ) such that $N \in \mathbb{Z}_{>0}$, V is a linear subspace of $H^0(X, \mathscr{L}^{\otimes N})$, and $\mu \in \mathbb{R}_{>0}$, with the following property. There is a finite collection $\mathscr{B}_1, \dots, \mathscr{B}_\ell$ of bases of V such that, for some (and hence all) choices of $v \in M_k$ and for some (and hence all) choices of Weil functions $\lambda_{\mathscr{B}_i,v}$ and $\lambda_{D,v}$ for $(\mathscr{B}_i)$ and D, respectively, for all i, such that

$$\max_{1\le i\le\ell} \lambda_{\mathscr{B}_i,v} \ge \mu N \lambda_{D,v} + O(1) \tag{1}$$

(as functions $X(\mathbb{C}_v) \to \mathbb{R} \cup \{+\infty\}$), where the implicit constant depends only on the data listed (and not on the point in $X(\mathbb{C}_v)$).

Definition. Let X be a complete variety, let $\mathscr{L}$ be a line sheaf on X, and let D be an effective $\mathbb{Q}$-Cartier divisor on X. Define

$$\operatorname{Nev}^{\mathrm{EF}}(\mathscr{L}, D) = \inf_{N,\mu} \frac{\dim X + 1}{\mu} ,$$

where the inf passes over all pairs (N, μ), with $N \in \mathbb{Z}_{>0}$ and $\mu \in \mathbb{R}_{>0}$, such that there is a set of pairs $(V_i, \mathscr{B}_i)$ ($i = 1, \dots, \ell$) such that

(i). for all i, V_i is a base-point-free linear subspace of $H^0(X, \mathscr{L}^{\otimes N})$ with $\dim V_i = \dim X + 1$;

(ii). for all i, $\mathscr{B}_i$ is a basis for V_i; and

(iii). for some (and hence all) $v \in M_k$ and for some choices of Weil functions $\lambda_{(\mathscr{B}_i),v}$ for $(\mathscr{B}_i)$ and $\lambda_{D,v}$ for D at v,

$$\max_{1 \le i \le \ell} \lambda_{(\mathscr{B}_i),v} \ge \mu N \lambda_{D,v} + O(1) \tag{2}$$

(as functions $X(\mathbb{C}_v) \to \mathbb{R} \cup \{+\infty\}$), where the implicit constant depends on everything but the point in $X(\mathbb{C}_v)$.