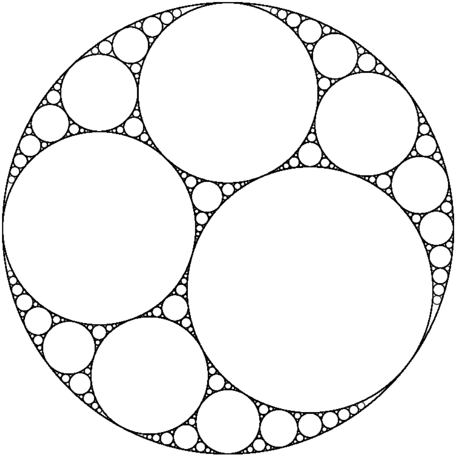
The Apollonian structure of imaginary quadratic fields

Katherine E. Stange

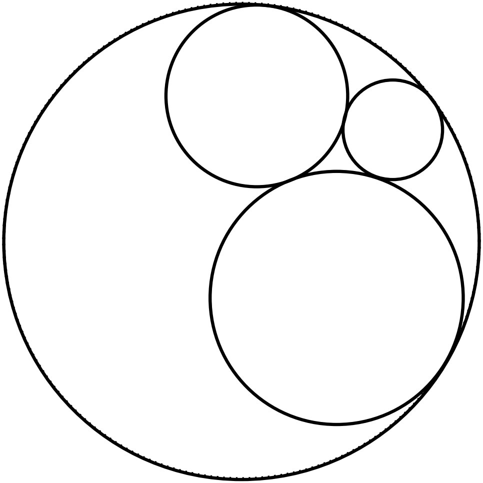
Silvermania, August 12, 2015

Apollonian Circle Packings



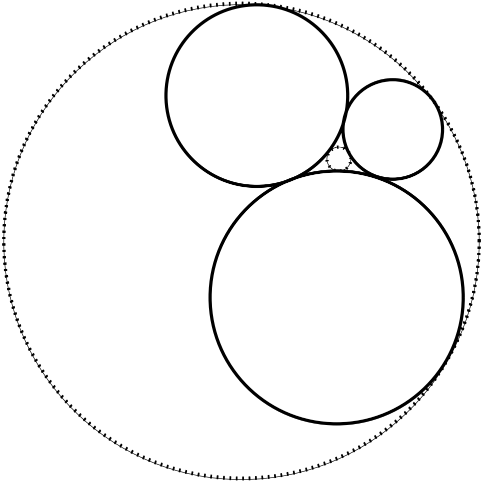
Apollonian Circle Packings

A Descartes quadruple is any collection of four circles which are pairwise mutually tangent, with disjoint interiors.



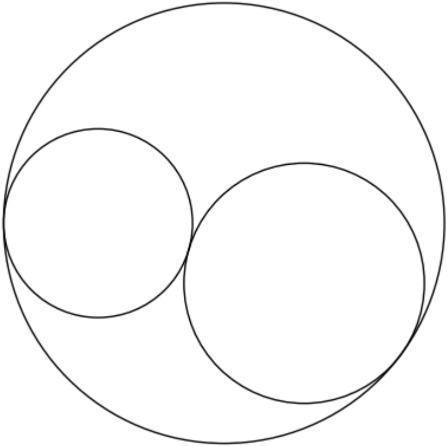
Apollonian Circle Packings

Given any three mutually tangent circles, there are exactly two ways to complete the triple to a Descartes quadruple.



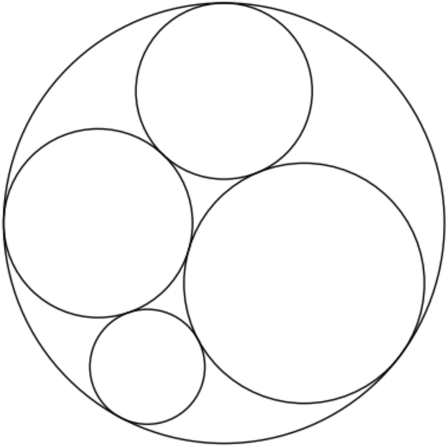
Apollonian Circle Packings

Beginning with any three mutually tangent circles...



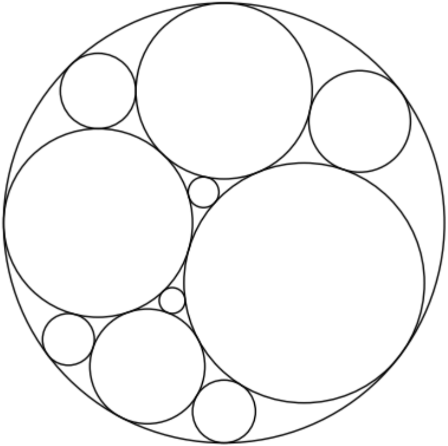
Apollonian Circle Packings

Beginning with any three mutually tangent circles, add in both new circles which would complete the triple to a Descartes quadruple.



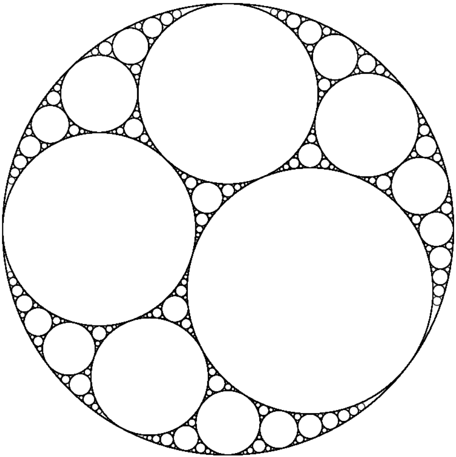
Apollonian Circle Packings

Repeat: for every triple of mutually tangent circles in the collection, add the two 'completions.'



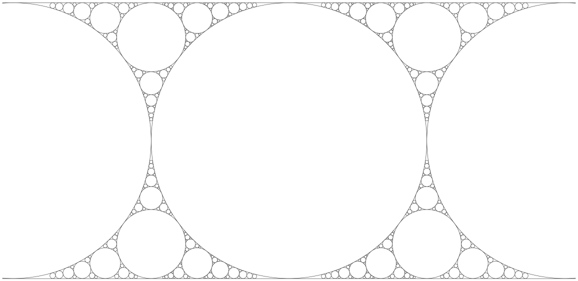
Apollonian Circle Packings

Repeating ad infinitum, we obtain an Apollonian circle packing.

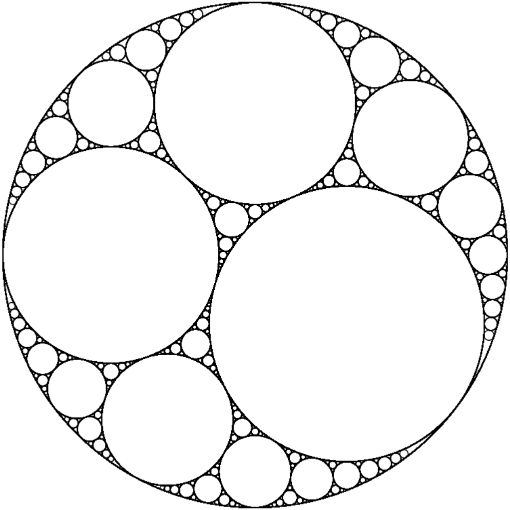


Apollonian Circle Packings

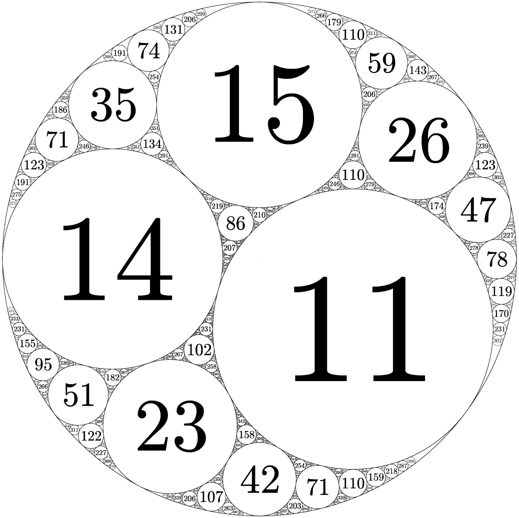
Repeating ad infinitum, we obtain an Apollonian circle packing.



Apollonian Circle Packings



Apollonian Circle Packings



The outer circle has curvature -6 (its interior is outside).

The Descartes Rule

The curvatures (inverse radii) in a Descartes configuration satisfy

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

If a, b, c are fixed, there are two solutions d, d', where

$$d + d' = 2(a + b + c).$$

Hence an **integer** Descartes quadruple generates an Apollonian packing of **integer curvatures.**

Local-Global Conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, Fuchs–Sanden)

$\mathcal{P}$ a primitive, integral ACP. Let S be the set of residues of curvatures modulo 24. Then any sufficiently large integer with a residue in S occurs as a curvature.

- Bourgain, Fuchs: Curvatures have positive density in $\mathbb{Z}$.
- Bourgain, Kontorovich: Density one occur.

Apollonian generalizations

Guettler, Mallows, *A generalization of Apollonian packing of circles*, J. of Comb., 2010.

4 Gerhard Guettler and Colin Mallows

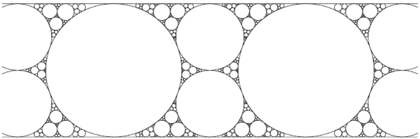
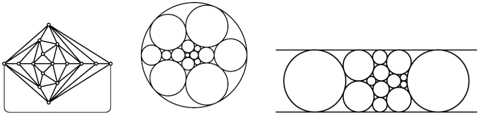


Figure 3: Another generalized Apollonian packing.

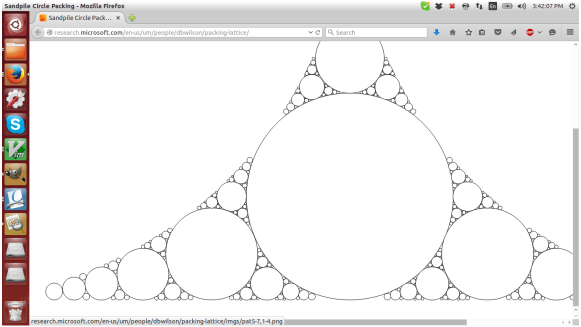
Butler, Graham, Guettler, Mallows, *Irreducible Apollonian Configurations and Packings*, Disc. Comp. Geom., 2010.



(a) Tangency graph (b) Generic packing (c) Standard packing

Fig. 2 Different representations of an Apollonian packing

Apollonian lattices



Thank you to David Wilson and Lionel Levine.

Schmidt Arrangements

The *Schmidt arrangement* of a imaginary quadratic field K is the orbit of $\widehat{\mathbb{R}}$ under the Möbius transformations given by the *Bianchi group*

$$\mathrm{PSL}_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} : \alpha, \beta, \delta, \gamma \in \mathcal{O}_K,\ \alpha\delta - \beta\gamma = 1 \right\} / \pm I$$

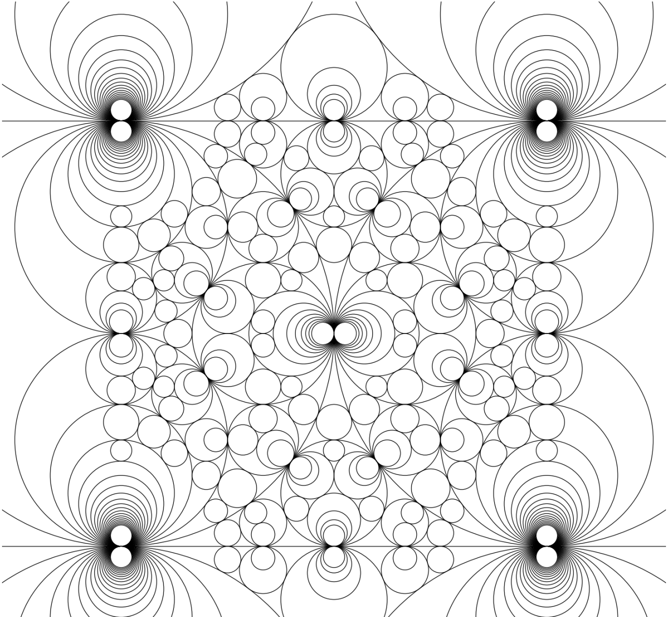
That is,

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \leftrightarrow \left(z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right).$$

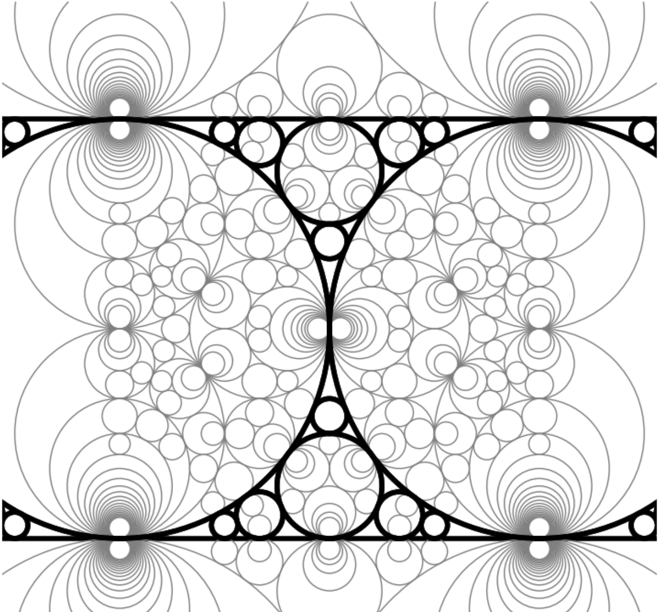
Each individual image $M(\widehat{\mathbb{R}})$ is called a *K-Bianchi circle*.

$$\mathcal{S}_K = \{K\text{-Bianchi circles}\}$$

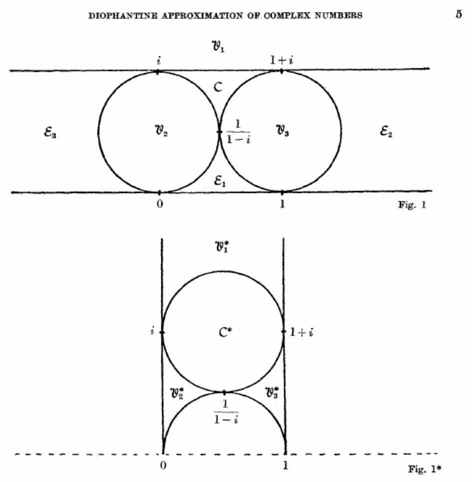
Schmidt Arrangement of $\mathbb{Q}(i)$



Schmidt Arrangement of $\mathbb{Q}(i)$



History: 1975



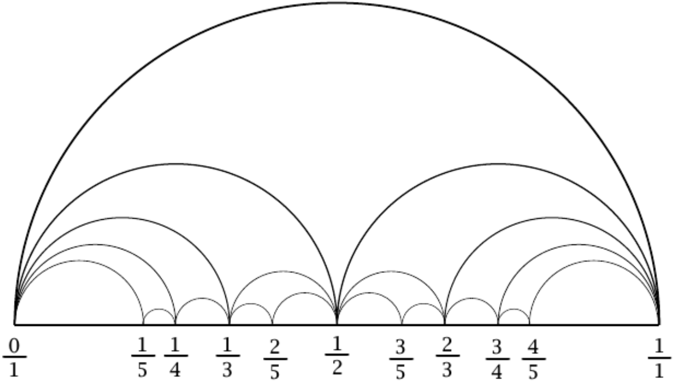
- Asmus Schmidt, *Diophantine Approximation of Complex Numbers*, Acta Arithmetica, 1975.
- Continued fractions for $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$ etc. made use of $\mathcal{S}_K$ (defined essentially this way).

Continued fractions

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}.$$

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

Farey division of the real line



Picture: Allen Hatcher, *The topology of numbers*.

History: 2006

Apollonian Circle Packings, II

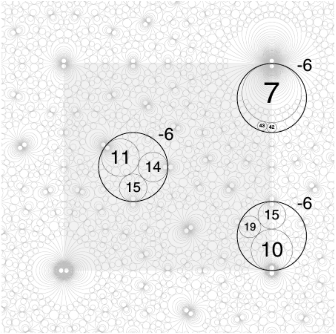
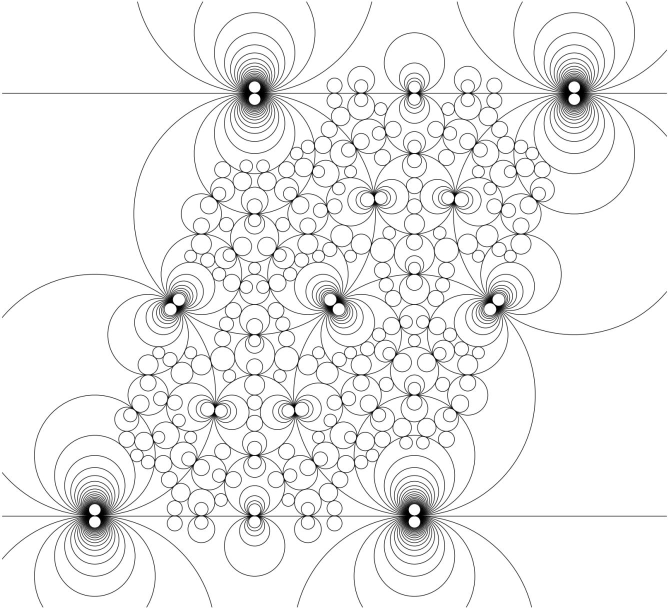


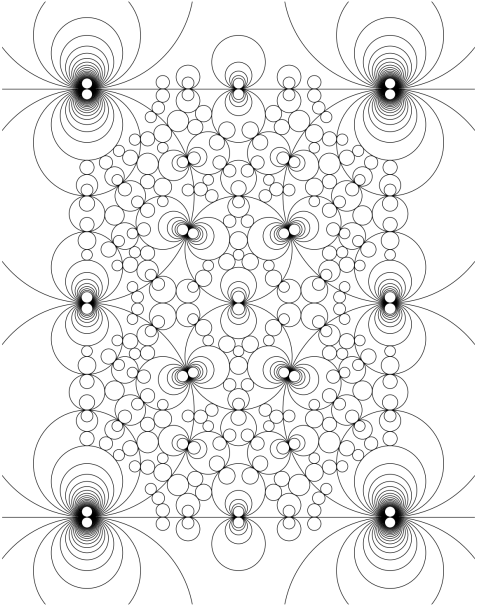
Fig. 5. Integer Apollonian packings with a bounding circle of curvature 6.

- Graham, Lagarias, Mallows, Wilks, Yan, *Apollonian Circle Packings: Geometry and Group Theory II. Super-Apollonian Group and Integral Packings*, Discrete and Computational Geometry, 2006.
- Superpacking defined differently but equivalently.

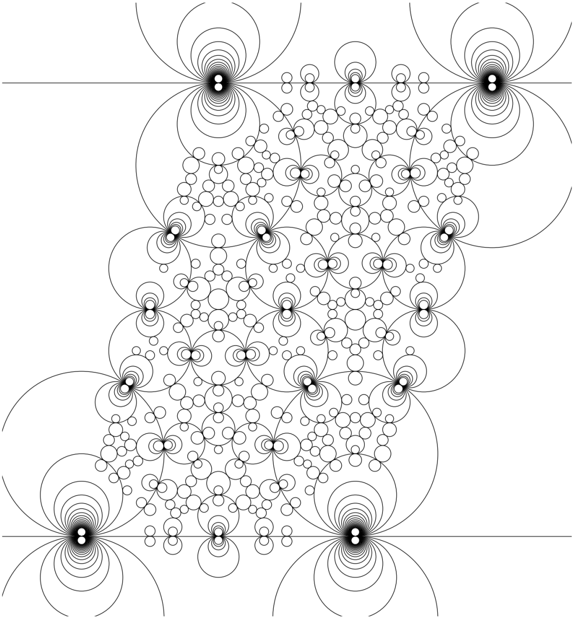
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-7})$



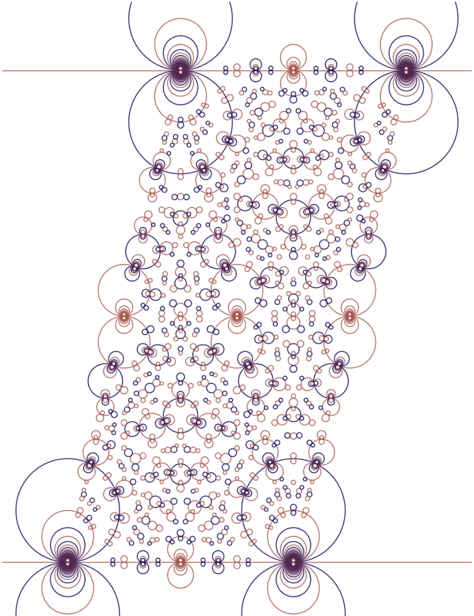
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-2})$



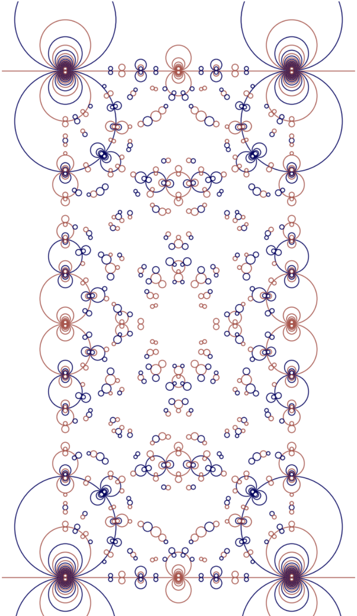
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-11})$



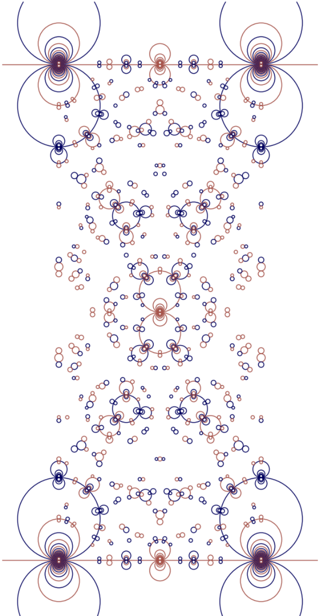
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-19})$



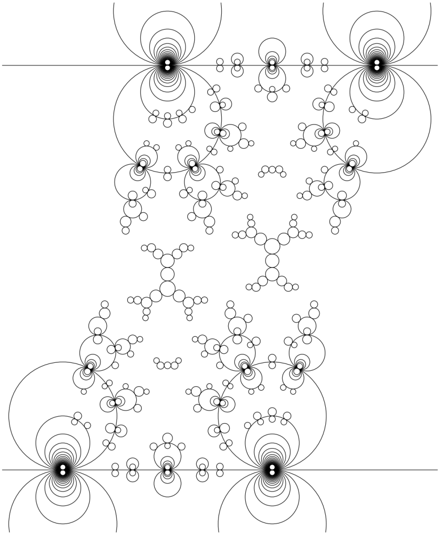
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-5})$



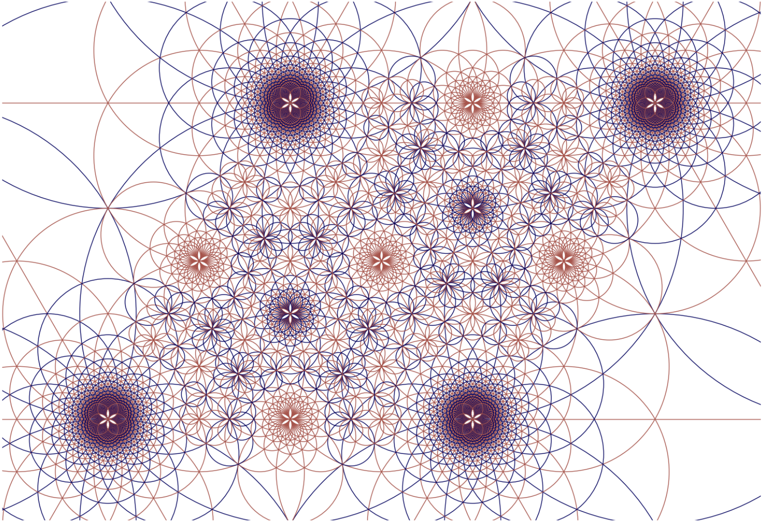
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-6})$



Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$



Schmidt Arrangement of $\mathbb{Q}(\sqrt{-3})$



Now the theme on AMS YouTube, Twitter, etc.

Basic properties of $\mathcal{S}_K$

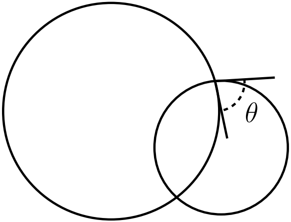
$$\Delta = \mathrm{Disc}(K)$$

Proposition (S.)

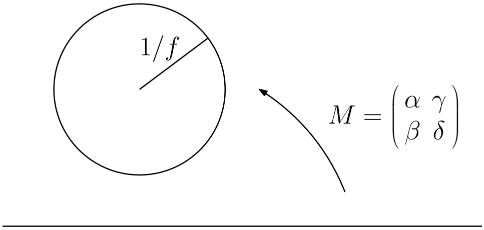
The curvatures in $\mathcal{S}_K$ lie in $\sqrt{-\Delta}\mathbb{Z}$.

Proposition (S.)

K-Bianchi circles intersect at points in K, at angles θ such that $e^{i\theta}$ is a unit in K.



Circles are ideal classes



Theorem (S.)

$$\left\{\begin{matrix}\textit{oriented}\\ \textit{circles}\end{matrix}\right\} \Big/ \left\{\begin{matrix}\textit{translations by } \mathcal{O}_K \textit{ and}\\ \textit{rotations by `unit angles'}\end{matrix}\right\} \qquad M(\widehat{\mathbb{R}}) \qquad f = \textit{curvature}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\left\{\begin{matrix}\textit{invertible}\\ \textit{ideal classes}\\ \mathfrak{a}\subset\mathcal{O}_f\end{matrix} \;\middle|\; f \in \mathbb{Z}^{>0}, \mathfrak{a}\mathcal{O}_K = \mathcal{O}_K\right\} \qquad \beta\mathbb{Z} + \delta\mathbb{Z} \qquad f = \textit{covolume}$$

Corollary: Number of circles of curvature f (up to equivalence) is h_f/h_K. (GLMWY for $\mathbb{Q}(i)$)

Euclideanity and $\mathcal{S}_K$

The *tangency graph* G_K of $\mathcal{S}_K$ is:

$$\left\{ \begin{array}{rcl} \textit{vertices} & = & \textit{circles} \\ \textit{edges} & = & \textit{tangencies} \end{array} \right\}.$$

Proposition (S.)

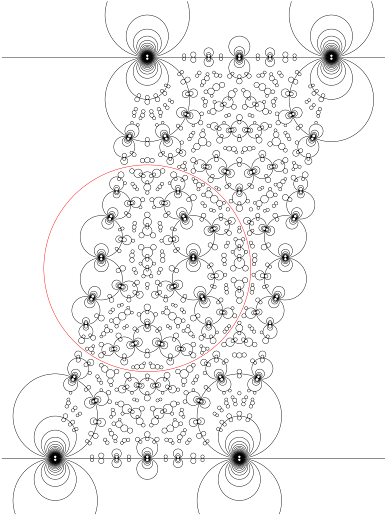
G_K is connected if and only if $\mathcal{O}_K$ is Euclidean.

Proof.

1. Connected component of $\widehat{\mathbb{R}}$ is all circles reachable by combinations of elementary matrices.
2. Thm of P.M. Cohn: $\mathcal{O}_K$ is Euclidean if and only if $\mathrm{SL}_2(\mathcal{O}_K)$ is generated by elementary matrices.

$\square$

Euclideanity and $\mathcal{S}_K$



Theorem (S.)

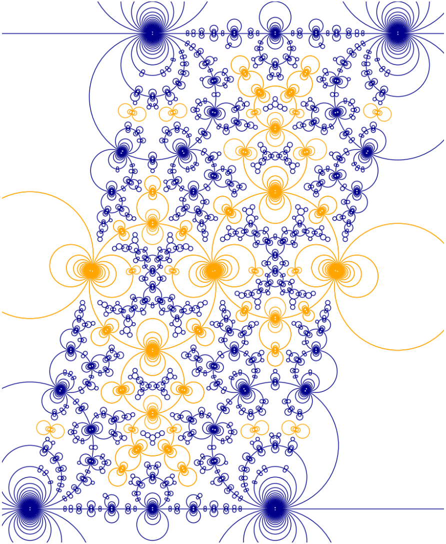
$\mathcal{S}_K$ is connected if and only if $\mathcal{O}_K$ is Euclidean.

The *ghost circle* is the circle orthogonal to the unit circle having center

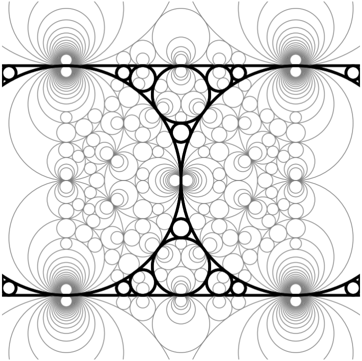
$$\begin{cases} \frac{1}{2} + \frac{\sqrt{\Delta}}{4} & \Delta \equiv 0 \pmod 4 \\ \frac{1}{2} + \frac{-\Delta - 1}{4\sqrt{\Delta}} & \Delta \equiv 1 \pmod 4 \end{cases}.$$

It exists only when $\mathcal{O}_K$ is non-Euclidean.

Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$ with Ghost Circles



Straddling

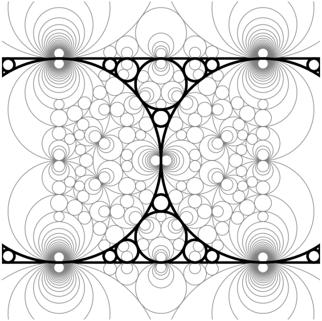


Definition

$\mathcal{P} \subset \mathcal{S}'_K$, $C, C_1, C_2 \in \mathcal{S}'_K$.

1. $\mathcal{P}$ *straddles* C if it intersects the interior and exterior of C
2. C_1 and C_2 are *immediately tangent* if they are externally tangent such that their union straddles no circle of $\mathcal{S}'_K$

K-Apollonian Packings

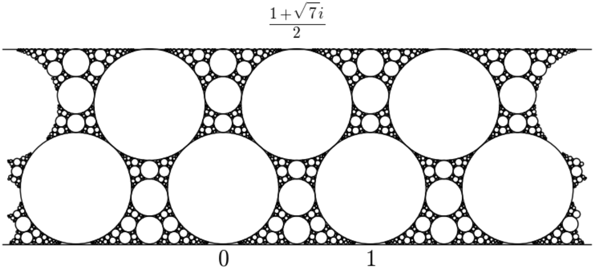


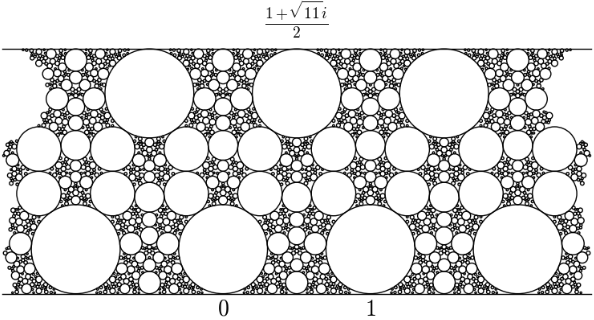
Theorem (S.)

The following are equivalent:

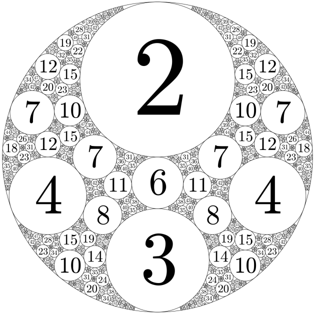
1. $\mathcal{P}$ *is a maximal tangency-connected set of circles with disjoint interiors and straddling no circle of* $\mathcal{S}'_K$
2. $\mathcal{P}$ *is a minimal non-empty set of circles that is closed under immediate tangency.*

K-Apollonian Packings





K-Apollonian Packings



Theorem (S.)

The Schmidt arrangement is the disjoint union of all K-Apollonian circle packings (where circles are oriented).

The exceptional isomorphism

$$\rho : \mathrm{PGL}_2(\mathbb{C}) \to \mathrm{SO}^+_{1,3}(\mathbb{R}).$$

- $\mathrm{SO}^+_{1,3}(\mathbb{R})$ acts on the 4D real vector space of Hermitian matrices,
$$\begin{pmatrix} b' & x+iy \\ x-iy & b \end{pmatrix}$$
preserving the determinant, which is a form of signature $3, 1$.
- $\mathrm{PGL}_2(\mathbb{C})$ acts by conjugation $\gamma \cdot M = \gamma^\dagger M \gamma$.
- Hermitian forms of determinant 1 (say) 'are' circles (take the zero set in $\widehat{\mathbb{C}}$). This is a hyperboloid in Minkowski space, a model of $\mathbb{H}^3$.

The Apollonian Group ($\mathbb{Z}[i]$)

Idea: act on *Descartes quadruples* instead of circles, coded as a 4×4 matrix

$$W_D = \begin{pmatrix} | & | & | & | \\ c_1 & c_2 & c_3 & c_4 \\ | & | & | & | \end{pmatrix}$$

Theorem (GLMWY)

C_1, C_2, C_3, C_4 form a Descartes configuration if and only if

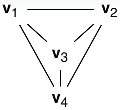
$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

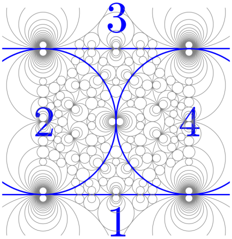
Codify swaps of Descartes quadruples as a matrix action:

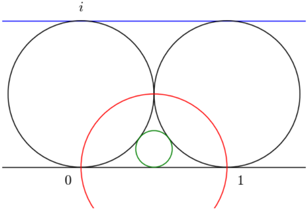
$$W_D \mapsto W_D S_i, \quad i = 1, 2, 3, 4$$

The *Apollonian group* is $\langle S_1, S_2, S_3, S_4 \rangle \subset \mathrm{O}_{3,1}(\mathbb{R})$.

Cheat Sheet for $\mathbb{Q}(i)$







$$W_D : \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$$

$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Apollonian group:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

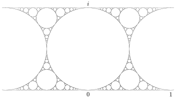
$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1 \rangle$$

The Apollonian Group

Codify swaps of Descartes quadruples as a matrix action:

$$W_D \mapsto W_D S_i, \quad i = 1, 2, 3, 4$$

The *Apollonian group* is $\langle S_1, S_2, S_3, S_4 \rangle$.

1. Freely generated by these four generators of order two.
2. *Thin*, i.e. infinite index in its Zariski closure $O_{3,1}(\mathbb{R})$.
3. Acts freely and transitively on the quadruples in a packing (so packing is orbit of 4 circles).
4. Limit set: 

5. Main tool in results on curvatures.

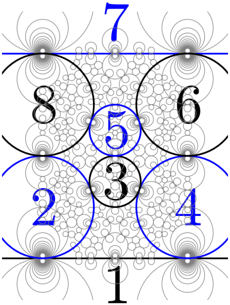
K-Apollonian groups

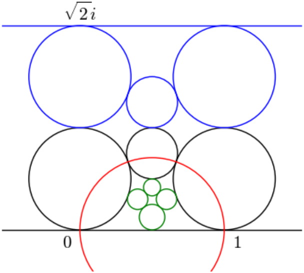
Theorem (S.)

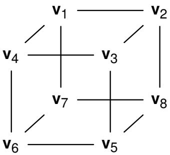
For each imaginary quadratic $K \neq \mathbb{Q}(\sqrt{-3})$, there is a Kleinian group $\mathcal{A} <$ Möb such that

1. *Its limit set is the K-Apollonian strip packing.*
2. *It acts freely and transitively on the clusters of any K-Apollonian packing (suitably defined).*
3. *It is finitely generated (with a simple presentation).*
4. *It is thin.*

Cheat Sheet for $\mathbb{Q}(\sqrt{-2})$







$$W_D : \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6, \mathbf{v}_8$$

$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -3 & -3 & -3 \\ -3 & 1 & -3 & -3 \\ -3 & -3 & 1 & -3 \\ -3 & -3 & -3 & 1 \end{pmatrix}$$

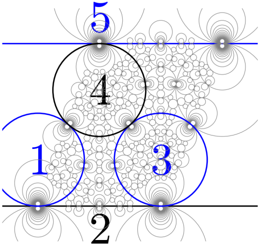
Apollonian group:

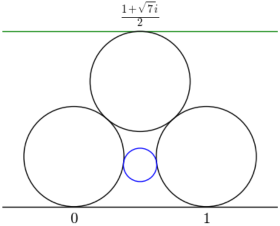
$$\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 3 & 1 & 0 & 3 \\ 3 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

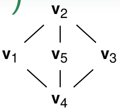
$$\begin{pmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 3 & 1 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{pmatrix}.$$

$$\langle r, s, t, u, v, w : r^2 = s^2 = t^2 = u^2 = v^2 = w^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{-7})$







$$W_D : \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$$

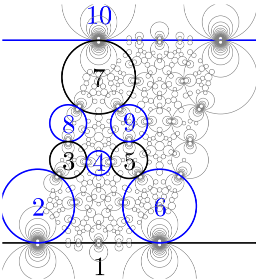
$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -1 & -5/2 & -1 \\ -1 & 1 & -1 & -5/2 \\ -5/2 & -1 & 1 & -1 \\ -1 & -5/2 & -1 & 1 \end{pmatrix}$$

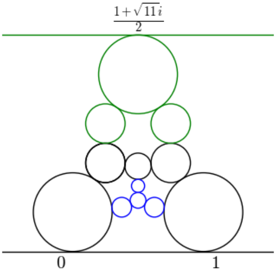
Apollonian group:

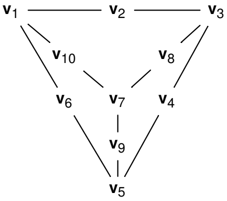
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 & -1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

$$\langle r, s, t : r^2 = s^2 = t^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{-11})$







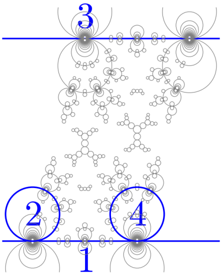
$W_D : \mathbf{v}_1 + \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_8, \mathbf{v}_1 + \mathbf{v}_9, \mathbf{v}_3 + \mathbf{v}_9$

Apollonian group:

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 3 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 3 & 3 & 3 & 1 \end{pmatrix}.$$

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{\Delta})$, $\Delta < -11$



Apollonian group ($\Delta \equiv 0 \pmod 4$):

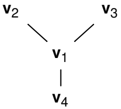
$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1+\frac{\Delta}{4} & 1 & 1+\frac{\Delta}{4} \\ 0 & 1 & 0 & 0 \\ 1 & -\frac{\Delta}{4}-1 & 0 & -\frac{\Delta}{4}-1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Apollonian group ($\Delta \equiv 1 \pmod 4$):

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \frac{\Delta+3}{4} & 0 \\ 1 & 1 & -\frac{\Delta-1}{4} & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -\frac{\Delta+3}{4} & 0 \end{pmatrix}.$$



- Swaps: swap out $\mathbf{v}_i$ for $i = 2, 3, 4$ or move $\mathbf{v}_1$ to $\mathbf{v}_2$.

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1, rstu = stur \rangle$$

Generalized Local-Global Conjecture

Conjecture (S.)

$\mathcal{P}$ a primitive, integral K-ACP for $K \neq \mathbb{Q}(\sqrt{-3})$ with discriminant Δ. Let S_M be the set of residues of curvatures modulo M. Then, for some $M \mid 24$, any sufficiently large integer with a residue in S_M occurs as a curvature.
A sufficient M is given by

$$v_2(M) = \begin{cases} 3 & \Delta \equiv 28 \pmod{32} \\ 2 & \Delta \equiv 8, 12, 20, 24 \pmod{32} \\ 1 & \Delta \equiv 0, 4, 16 \pmod{32} \\ 0 & \textit{otherwise} \end{cases},$$

$$v_3(M) = \begin{cases} 1 & \Delta \equiv 5, 8 \pmod{12} \\ 0 & \textit{otherwise} \end{cases}.$$