The arithmetic dynamics of correspondences

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Arithmetic dynamics

(Arithmetic) dynamics

Let K be a (number) field, X/K a variety, and $f : X \to X$ a morphism. Describe

$$\mathscr{O}_f^+(P) = \{P, f(P), f^2(P) = f \circ f(P), \ldots\}$$

and maybe

$$\mathscr{O}_f^-(P) = \{P, f^{-1}(P), f^{-2}(P), \ldots\}.$$

Size of orbit, convergence, local behaviour at fixed points, behaviour of critical points, etc...

The canonical height

If X is projective, L is an ample $\mathbb{R}$-divisor, and $f^*L \sim \alpha L$ for some real $\alpha > 1$, the Call-Silverman canonical height satisfies

$$\hat{h}_{X,L,f}(P) = h_{X,L}(P) + O(1)$$
$$\hat{h}_{X,L,f}(f(P)) = \alpha \hat{h}_{X,L,f}(P)$$
$$\hat{h}_{X,L,f}(P) = 0 \quad \Leftrightarrow \quad P \text{ has finite orbit.}$$

In particular, if $K = \mathbb{Q}$ and $f^n(x) = a_n/b_n$, then

$$\log \max\{|a_n|, |b_n|\} = d^n \hat{h}_f(x) + O(1).$$

Correspondences

Many-valued dynamics

Rather than iterate

$$y = x^3 + 1,$$

what if we iterate

$$y^2 = x^3 + 1?$$

The path space

Let X/K be a variety ($\mathbb{P}^1$ for most of this), and let $C \subseteq X^2$ have both coordinate projections finite and surjective.

There exists a K-scheme $\pi : \mathscr{P} \to X$ and a finite morphism $\sigma : \mathscr{P} \to \mathscr{P}$ such that...

$$\begin{tikzcd}
\mathscr{P} \arrow[r] \arrow[d, "\pi"'] & {} \arrow[r, "\sigma"] \arrow[d, "\epsilon"] & \mathscr{P} \arrow[d, "\pi"] \\
X & C \arrow[l, "x"] \arrow[r, "y"'] & X
\end{tikzcd}$$

$\mathscr{P}$ parametrizes paths defined by iterating the correspondence C, starting at the point marked by π.

The path space

If C is the graph of a morphism f, then $\mathscr{P} \cong X$ with $\sigma = f$.

If $C : x = f(y)$, then $\mathscr{P} \to X$ describes "inverse image trees."

In general, you can think of $\pi^{-1}(x) \subseteq \mathscr{P}$ as a tree, a probability space, and/or a totally disconnected compact Hausdorff space.

In some cases this is easy to construct. For instance, if $X = \mathbb{A}^1$ and $C : F(x, y) = 0$, then $\mathscr{P} = \mathrm{Spec(R)}$ with

$$R = K[x_0, x_1, ...]/(F(x_i, x_{i+1}) : i \geq 0).$$

An annoyance

$\sigma : \mathscr{P} \to \mathscr{P}$ is an algebraic dynamical system encapsulating the correspondence, but $\mathscr{P}$ is not in general a variety.

The property $X(\overline{K}) = \bigcup_{[L:K]<\infty} X(L)$ of varieties is quite useful!

Let $C : y^2 = x^3 + 1$. For $P \in \mathscr{P}(K)$, we can make S large enough so that P is supported on S-integral points. This means that P is finitely supported.

We have $\bigcup_{[L:K]<\infty} \mathscr{P}(L)$ consisting in just finitely supported paths... but this is certainly not the case for the typical element of $\mathscr{P}(\overline{K})$.

The canonical height

Polarized correspondences

Now assume X is projective.

We say that C is *polarized* if there is an ample $L \subseteq \mathrm{Pic(X)} \otimes \mathbb{R}$ and a real $\alpha > 1$ with $y^*L \sim \alpha x^*L$.

With $X = \mathbb{P}^1$ and $C : g(y) = f(x)$, the condition comes down to $\deg(g) < \deg(f)$.

A canonical height

Theorem (I. 2014)

Given a polarized correspondence, there exists a $\hat{h}_{X,L,C} : \mathscr{P}(\overline{K}) \to$ *such that...*

1. $\hat{h}_{X,L,C}(P) = h_{X,L} \circ \pi(P) + O(1)$
2. $\hat{h}_{X,L,C} \circ \sigma(P) = \alpha \hat{h}_{X,L,C}(P)$
3. $\hat{h}_{X,L,C}(P) = 0$ *if* P *is finitely supported.*

The converse to the last claim holds true on

$$\bigcup_{[L:K]<\infty} \mathscr{P}(L),$$

but this is generally a small subset of $\mathscr{P}(\overline{K})$.

Comments on the canonical height

Call $x \in X(\overline{K})$ *constrained* if there exists a finitely supported path P with $\pi(P) = x$ (i.e., if the orbit of x is not an honest tree).

As a corollary to the above, the set of constrained points is a set of bounded height.

Comments on the canonical height

Note that for $C : y^2 = x^3 + 1$ we have

$$\bigcup_{[L:K]<\infty} \mathscr{P}(L) \subseteq \{P \in \mathscr{P}(\overline{K}) : \hat{h}(P) = 0\}.$$

Of course, those are all finitely supported paths.

If $\hat{h}(P) = 0$ and $P \in \mathscr{P}(L)$ for some $[L : K] < \infty$, then P is finitely supported.

On the other hand, every path P for $y^2 = x^3$ with $\pi(P) = -1$ has $\hat{h}(P) = 0$, and none is finitely supported.

The restriction to fibres

Note that for each $a \in X$, $\pi^{-1}(a) \subseteq \mathscr{P}$ is naturally a compact Hausdorff space under the tree topology, with a Borel probability.

Theorem (I. 2014)

For any $a \in X(\overline{K})$, $\hat{h}_{X,L,C}$ is continuous and measurable on $\pi^{-1}(a)$. In particular,

$$min_{\pi(P)=a}\hat{h}_{X,L,C}(P) \leq \mathbb{E}(\hat{h}_{X,L,C}(P)|\pi(P)=a) \leq max_{\pi(P)=a}\hat{h}_{X,L,C}(P)$$

all make sense.

Note: Autissier's canonical height for correspondences turns out to be the middle thing.

Local heights

Recall that the height of $\alpha \in K$ is defined by

$$h(\alpha) = \sum_{v \in M_K} \log^+ |\alpha|_v \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}.$$

Working over $\overline{K}$ introduces some difficulties.

Gubler introduces a measure μ on $M_{\overline{K}}$ such that

$$h(\alpha) = \int_{M_{\overline{K}}} \log^+ |\alpha|_v d\mu(v).$$

Local heights

Theorem (I. 2014)

There exist local height functions $\lambda_{X,L,C} : \mathscr{P} \times M_{\overline{K}}$ such that

$$\hat{h}_{X,L,C}(P) = \int_{M_{\overline{K}}} \lambda_{X,L,C}(P, v) d\mu(v)$$

for $P \notin \mathrm{Supp}(L)$.

Note that "local height function" needs to be re-defined in order to make sense on something that's not a variety!

Specialization

Theorem (Silverman 1983?)

For a section P of an elliptic surface $\mathcal{E} \to B$, we have

$$\hat{h}_{\mathcal{E}_t}(P_t) = \left(\hat{h}_{\mathcal{E}}(P) + o(1)\right) h_B(t)$$

where $o(1) \to 0$ as $h_B(t) \to \infty$.

Call-Silverman proved the analogue for families of dynamical systems.

Specialization

Theorem (I. 2014)

For a family of correspondences C on $X \to B$, and a path P with $\pi(P) : B \to X$, we have

$$\hat{h}_{C_t}(P_t) = \left(\hat{h}_C(P) + o(1)\right) h_B(t)$$

For instance, if $\hat{h}_C(P) > 0$, the set of $t \in B$ with P_t finitely supported is a set of bounded height.

Thank you.

Critical orbits

In single-valued dynamics, the orbits of critical points are (unsurprisingly) important.

A morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is PCF if and only if its critical points all have finite (forward) orbit.

Conjecture (Silverman 2010)

$$h_{M_d}(f) \gg\ll h_{\mathrm{Crit}}(f) := \sum_{c \in \mathrm{Crit(f)}} \hat{h}_f(c),$$

once Lattés maps are excluded.

Critical orbits

Theorem (I. 2011, 2013)

This is true for polynomials on $\mathbb{P}^1$, and for a class of maps generalizing polynomials on $\mathbb{P}^N$. In fact,

$$h_{M_d}(f) = h_{\mathrm{Crit}}(f) + O(1)$$

if you completely re-define both sides.

Theorem (Benedetto-I.-Jones-Levy 2014)

The PCF points form a set of bounded height in the moduli space M_d of rational functions of degree $d \geq 2$, once Lattés examples are excluded.

PCC

A *critical point* for the correspondence C will be the x-coordinate of any point at which x or y ramifies.

Call C *post-critically constrained* (PCC) iff for every $c \in \mathrm{Crit(C)}$, there exists a finitely supported $P \in \mathscr{P}$ with $\pi(P) = c$.

E.g., $y^2 = x^d + 1$ whenever d is odd.

Critical height

Theorem (I. 2014)

For $C : g(y) = f(x)$, with g, f polynomials,

$$h_{\mathrm{Weil}}(C) = h_{\mathsf{Crit}}(C) + O(1).$$

Theorem (I. 2014)

Over $\mathbb{C}$, with setup as above, the correspondences for which every critical point admits a bounded path form a compact subset of moduli. space.

Theorem (I. 2014)

In residue characteristic 0 or $p > d$, there are no algebraic families of PCC correspondences of the above form.

Thank you.

The action of Galois

Arboreal Galois representations

For $f(z) \in K(z)$ and $x \in K$, define $T \approx \mathcal{O}_f^-(x)$ to be the preimage tree. Consider

$$\rho_{f,x} : \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K}) \to \mathrm{Aut}(\mathrm{T})$$

by the action on nodes in the tree

When is this (nearly) surjective?

Expanding the arboretum

Let C be a correspondence on X, defined over K, and let $\pi : \mathscr{P} \to X$ be the space of paths.

Since $\mathscr{P}$ is a K-scheme, there is a natural action of $G = \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ on $\pi^{-1}(x) \subseteq \mathscr{P}(\overline{K})$ for any $x \in X(K)$.

The graph structure on $\pi^{-1}(x)$ is K-rational, and so we have

$$\rho_{C,x} : G \to \mathrm{Aut(T)},$$

where T is $\pi^{-1}(x)$ as a directed graph (which might not be a tree!!).

The image of Galois

It is natural to ask when $\rho_{C,x}$ is (nearly) surjective.

Conjecture (Automatic generalization of folklore)

The image of $\rho_{C,x}$ has finite index in $\mathrm{Aut}(\mathrm{T})$*, except for sometimes.*

The conjecture is true (but stupid) for $C : y = f(x)$ (forward orbits).

Jones, Hindes have proven various cases for $C : x = f(y)$ (backward orbits).

Some kind of result

Theorem (I. 2014)

Let K be a complete, non-archimedean field, let $f, g \in K[x]$ have good reduction and $\deg g < \deg f$ both relatively prime to the residue characteristic of K, and let $C : g(y) = f(x)$. Then there is a Galois-equivariant bijection between

$$\{P \in \mathscr{P}(\overline{K}) : |\pi(P)| > 1\}$$

and the corresponding set for $y^{\deg(g)} = x^{\deg(f)}$.

Kummer theory then gives some description of the action of Galois.

This action is much smaller than one would hope, though, over a number field, especially when $\gcd(\deg(f), \deg(g)) > 1$.