(0)

Brauer–Siegel ratio for abelian varieties over global fields.

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August 13, SILVERMANIA = conference for the 60th Birthday of Joe SILVERMAN.

j.w with

Ref: MH: Why is it difficult to compute Mordell-Weil group, Proceeding Scuola Normale Sup 2007

A. Pacheco, M.H. An analogue of the Brauer–Siegel theorem for ab. var. with char > 0
To appear in Moscow. J. M. (2015?)

second title: how large should be the generators of the MW group.

A/K A = abelian variety of dimension d

K = global field = $\begin{cases} \text{number field} \\ \mathbb{F}_q(C) \end{cases}$

Tentative answers: $A(K) = \underbrace{A(K)_{torsion}}_{finite} \oplus \mathbb{Z}P_1 \oplus \cdots \oplus \mathbb{Z}P_r$

① $\hat{h}(P_1) \ldots \hat{h}(P_r) \ll_\varepsilon H(A/K)^{1+\varepsilon}$

② can it be much smaller?

(for an elliptic curve we may think $H(A) = \max(|c_4|^{1/4}, |c_6|^{1/6})$,

Note 1: lower bounds for the minimum of non zero $\hat{h}(P)$ are "easy algorithmically" and there a few theoretical results (e.g Silverman – H.)

Note 2: the height $\hat{h}$ actually depends on a polarisation.

Classical Brauer - Siegel

$[K:\mathbb{Q}]=d \quad \Delta_K \to \infty$

then
$$\lim_K \frac{\log(h_K R_K)}{\log\sqrt{\Delta_K}} = 1$$

h_K = class number
R_K = regulator of units in $\mathcal{O}_K^\times$

(variant $\Delta_K^{\frac{1}{2}-\varepsilon} \ll h_K R_K \ll \Delta_K^{\frac{1}{2}+\varepsilon}$)

example $K=\mathbb{Q}(\sqrt{-D})$ then $h_D \approx \sqrt{D}$

$K=\mathbb{Q}(\sqrt{D})$ then $h_D \log \varepsilon_D \approx \sqrt{D}$

Note: a) there exist real fields with small unit $\log\varepsilon_D \approx \log D$ and hence large h_D
But it is unknown if there are (infinitely many) D's with small h_D.

b) Easily $\log\varepsilon_D \gg \log D$ (i.e. "Lang's conjecture is true in this context)

«Proof» i) class # formula

$$\lim (s-1)\zeta_K(s) = \frac{h_K R_K}{\sqrt{\Delta_K}} \cdot \frac{2^{r_1}(2\pi)^{r_2}}{\#\mathbb{G}_m(\mathcal{O}_K)_{\text{torsion}}}$$

ii) $$\Delta^{-\varepsilon} \ll \zeta^*(1) \ll (\log\Delta)^{d-1}$$

(if GRH true $\Delta/\log\Delta$)

Recall:

Hermite Minkowsky + Hadamard. $\Lambda \subset (E, \|\cdot\|)$ $\quad \Lambda \simeq \mathbb{Z}^n$, $E \simeq \mathbb{R}^n$ $\quad \mathcal{B}=\{e_1,\dots,e_n\}$ basis

$$\underset{(\forall \mathcal{B})}{\mathrm{vol}(\Lambda) \leq} \|e_1\| \cdots \|e_n\| \underset{\exists\mathcal{B}}{\leq} c_n \mathrm{vol}(\Lambda)$$

Abelian varieties

$\left.\begin{array}{l} K \text{ global} \\ d = \dim A \end{array}\right\}$ fixed

②

Néron-Tate regulator.

$$\langle \cdot, \cdot \rangle : A(K) \times \check{A}(K) \longrightarrow \mathbb{R}$$

$$\mathrm{Reg}(A/K) = |\det \langle P_i, \check{P}_j \rangle|$$

if L is a principal polarisation, this is (up to factor 2^r) the Gram determinant associated to $\hat{h}_L$

- analogy $R_K \longleftrightarrow \mathrm{Reg}(A/K)$
- analogy $h_K \longleftrightarrow \#Ш(A/K)$ (Shafarevich-Tate group)

  1st argument the BSD formula

  2nd argument

copy the Galois cohomology sequences leading to Ш ie

$$0 \longrightarrow A(K) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow H^1(G_K, A_{tor}) \to H^1(G_K, A) \to 0$$

$$0 \longrightarrow A(K) \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{Sel}(A/K) \to Ш(A/K) \to 0$$

we obtain from $0 \to \mu_n \to \mathcal{O}_{\bar{K}}^\times \to \mathcal{O}_{\bar{K}}^\times \to 0$ and the limit.

$$0 \to \mathcal{O}_K^\times \otimes \mathbb{Q}/\mathbb{Z} \to H^1(G_K, \mu) \to H^1(G_K, \bar{\mathbb{Z}}^\times) \to 0$$

$$0 \to \mathcal{O}_K^\times \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{Sel}(\mathcal{O}_K^\times, K) \to Ш(\mathcal{O}_K^\times, K) \to 0$$

$$\text{«exercise»} \longrightarrow \cong \text{ Class group of } K.$$

- analogy $\sqrt{\Delta_K} \longleftrightarrow H(A)$
- the term $\#\left(\begin{array}{c}\text{root of unity} \\ \text{of } K\end{array}\right) = \#\mathbb{G}_m(\mathcal{O}_K)_{tor}$

is written that pedantic way to suggest analogy with $\#(A(K) \times \check{A}(K))_{tor}$

Faltings height

$\mathcal{A}$ Néron model

$\downarrow \uparrow e$ neutral section.

$B = \begin{cases} \mathrm{spec}(\mathcal{O}_K) \\ \text{or} \\ C \end{cases}$

$\omega_A := e^* \Omega^d_{\mathcal{A}/B}$ line bundle on B

in the number field case we need metrics:

$$\|\alpha\|^2 = (2\pi)^{-d} \int_{A(\mathbb{C})} |\alpha \wedge \bar{\alpha}|$$

$$h(A/K) = \deg_C \omega_A \quad (\text{f.f. case})$$

$$\frac{1}{[K:\mathbb{Q}]} \widehat{\deg}(\omega_A, \|\,\|) \quad (\text{n.f. case})$$

if $A/\mathbb{Q}$ and α generator over $\mathbb{Z}$ of $\Omega^d_{\mathcal{A}/\mathbb{Z}}$ then

$$h(A/\mathbb{Q}) = -\frac{1}{2} \log (2\pi)^{-d} \int_{A(\mathbb{C})} |\alpha \wedge \bar{\alpha}|.$$

<u>define</u> $H(A) = \exp h(A/K)$ (n.f. case)

$q^{\deg(\omega_A)}$ (f.f case)

<u>lemma</u> (n.f. case) $H(A) \ll \Omega_A^{-1} \ll H(A)\, h(A)^{d/2} \ll H(A)^{1+\varepsilon}$

Notation $\tilde{H}(A/K) = \Omega_A^{-1}$ (n.f case)

$H(A)$ (f.f case).

<u>BSD conj</u>

3) $$\lim (s-1)^{-r} L(A,s) = \frac{\#Ш \cdot \mathrm{Reg}(A/K)}{\tilde{H}(A/K)} \frac{\prod c_v(A)}{\# A \times \check{A}(K)_{\mathrm{tor}}}$$

2) Ш finite

1) $r = \mathrm{ord}_{s=1} L(A,s)$

observations

1) analogy with class number formula.
2) Very partial results over n.f.
   situation better over f.f.

(Tate – Milne – ...... – Kato-Trihan):

- $r \leq \operatorname{ord}_{s=1} L(A,s)$
- equality ⟺ Ш finite ⟺ $\exists \ell$ Ш$[\ell^\infty]$ finite
- if ① true, the full BSD is true.

first conclusion instead of looking for the size of $\mathrm{Reg}(A/K)$ and of Ш(A/K),

we should look at the size of

$$|Ш(A/K)|\,\mathrm{Reg}(A/K) \quad \text{which is} \quad L^*(A,1)\,\frac{|A\times \check{A}(K)_{tor}|\,\widehat{H}(A)}{T(A)}$$

Theorem (AP–MH)

$$1 \leq |A\times\check{A}(K)_{tor}| \ll h(A)^{2d} \ll H(A)^{\varepsilon}$$
$$1 \leq T(A) \ll H(A)^{\varepsilon}$$

here $T(A)=\prod c_v(A)$ is the Tamagawa number

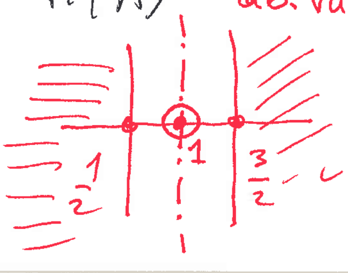
in fact $$\sum_{v \text{ bad}} c_v(A/K)^{1/d} \log q_v \ll h(A)$$

Theorem (assume "standard conjectures" in the n.f. case)
"analytic continuation + f.e + GRH"

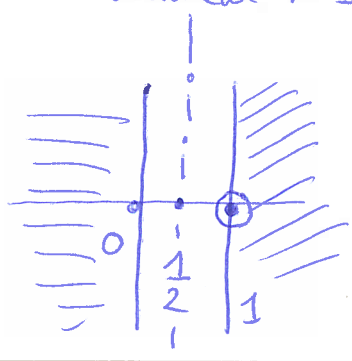
$$L^*(A,1) \ll N_A^{\varepsilon} \ll H(A)^{\varepsilon}$$

N_A conductor

BS for ab. var.



classical BS



2nd Conclusion

(f.f. case) assume Ш finite then

$$|Ш|\,\mathrm{Reg}(A/K) \ll H(A)^{1+\varepsilon} \qquad (*)$$

(n.f. case) assume BSD + ... + GRH then the same is true.

if we prefer estimates in terms of the conductor.
we invoke a (generalized) conjecture of Szpiro

¿for A/K semi-stable, $h(A/K) \leq \left(\frac{d}{2}+\varepsilon\right)\log N_A + C_\varepsilon$?

for (f.f.) the following is true:

assume [Kodaira-Spencer nonzero + Trace zero]

then $h(A) \leq \frac{d}{2}\left(2g(C) - 2 + \underbrace{s}_{\text{places of bad reduction}}\right)$

3rd conclusion: to discuss the optimality of (*)

introduce

$$BS(A/K) = \frac{\log |Ш|\,\mathrm{Reg}(A/K)}{\log H(A)}.$$

the classical Brauer-Siegel can be restated

as $\lim\limits_{\Delta_K \to \infty} BS(K) = 1$

statement (*) means $\limsup\limits_{H(A)\to\infty} BS(A/K) \leq 1$

Example of families

(1) let $E_m : y^2 + xy = x^3 - t^m$ over $\mathbb{F}_p(t)$

(say $p \nmid m$, $p \neq 2, 3$)

Thm a) Ш finite (Ulmer)

b) $\lim BS(E_m) = 1$ (H.-Pacheco)

in fact $\log Ш \operatorname{Reg}(E_m) \sim \frac{m}{6} \log q$.

Thm (Griffon) let $C_m : (t^m + 1) y^2 = x^3 + ax + b$

$m \in S = \{m$ such that there exists n with $m \mid p^n + 1\}$

a) Ш finite (Milne)

b) $\lim_{m \in S} BS(C_m / \mathbb{F}_p(t)) = 1$ (Griffon)

Thus we see that, taking the family of all. ab. var.

$$0 \leq \liminf BS(A/K) \leq \limsup BS(A/K) = 1$$

$\uparrow$

because $\operatorname{Reg}(A/K) \gg H(A)^{-\varepsilon}$

So the question is the true value of the liminf $\in [0, 1]$

two reasons to believe : $\liminf_{A} BS(A) = 0$? 7

① Twist of an elliptic curve (no. f and f. f)

$E : y^2 = x^3 + ax + b$ $E_D : Dy^2 = x^3 + ax + b$.

Waldspurger (analog over $\mathbb{F}_q(t)$ by Altuğ–Tsimerman)

$$L(E_D, 1) = K \frac{c(|D|)^2}{\sqrt{|D|}},$$

where $c(|D|)$ is the Fourier coefficient of a weight $\frac{3}{2}$ cuspidal modular form so we expect:

$$\limsup \frac{\log|c(|D|)|}{\log|D|} = \frac{1}{4}$$ (True over f. f / Ramanujan conjecture over n.f.)

$$\liminf_{c(|D|)\neq 0} \frac{\log|c(|D|)|}{\log D} \overset{??}{=} 0$$ (computational evidence + analogies)

② Twist of a constant elliptic curve. (over $\mathbb{F}_q[t]$)

$E_D : Dy^2 = x^3 + ax + b$ $a, b \in \mathbb{F}_q$

(D squarefree in $\mathbb{F}_q[t]$)

Milne: Ш(E_D) finite and BSD true.

define C_D the curve $v^2 = D(u)$

let $L(E/\mathbb{F}_q, T) = (1-\alpha T)(1-\bar{\alpha} T) = 1 - aT + qT^2$

$$L(C_D, T) = \prod_{j=1}^{2g} (1 - \beta_j T)$$

(Weil polynomial of curves)

Then (Milne) + easy computation

$$L^*(E_{D/K}, 1) = (\log q)^r \prod_{1\leq j\leq 2g}^{*} (1-\beta_j \alpha^{-1})(1-\beta_j \bar{\alpha}^{-1})$$

(* means that zero terms are omitted)

$$= (\log q)^r \left| L^*(C_D, \alpha^{-1}) \right|^2$$

Using the functional equation $L(T) = q^g T^{2g} L\left(\frac{1}{qT}\right)$

and $L^*(T) = q^{g-r} T^{2g-r} L^*\left(\frac{1}{qT}\right)$

we may write

$$L(T) = T^g G_D(T^{-1} + qT)$$

and

$$L^*(E_D, 1) = (\log q)^r \frac{G_D^*(a)^2}{q^{g-r}}$$

here $H(E_D) = q^{g+1}$ (the square is linked to the fact that $\#Ш$ is a square)

G_D (resp G_D^*) is a polynomial in $\mathbb{Z}[T]$ with all its roots real in $[-2\sqrt{q}, 2\sqrt{q}]$

a is an integer in the same interval.

a natural guess is that when D varies and a varies then $G_D^*(a)$ assume small values (non zero)