Lower bounds for average values of Arakelov-Green functions and global applications

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Lang's Conjecture

In their 1988 Inventiones paper, Hindry and Silverman proved the function field case of Lang's conjectural lower bound on the canonical height of non-torsion rational points:

Lang's Conjecture: Let K be a number field. There is a constant $c = c(K) > 0$ so that for all elliptic curves E/K and all non-torsion points $P \in E(K)$,

$$\hat{h}_E(P) \geq c \log N_{K/\mathbb{Q}} \mathcal{D}_{E/K}.$$

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Theorem (Hindry–Silverman)

1. *If K is a one-dimensional function field of characteristic 0, then the analogue of Lang's conjecture over K is true.*
2. *If K is a number field, then Szpiro's conjecture implies Lang's conjecture over K.*

Lang's conjecture is still open in the number field case.

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Szpiro's conjecture

Let K be a number field. If E/K is an elliptic curve, define the *Szpiro ratio* of E to be

$$\sigma_{E/K} = \frac{\log N_{K/\mathbb{Q}} \mathcal{D}_{E/K}}{\log N_{K/\mathbb{Q}} \mathcal{F}_{E/K}}.$$

Szpiro's Conjecture: Let K be a number field. For any $\varepsilon > 0$, there are only finitely many elliptic curves E/K such that

$$\sigma_{E/K} \geq 6 + \varepsilon.$$

In particular, there is a constant $M = M(K)$ such that $\sigma_{E/K} \leq M$ for all E/K.

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1. Szpiro's conjecture is equivalent to the *ABC-conjecture*.

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Petsche's quantitative refinement

Following Petsche (2006), we will sketch a proof of the following result.

Theorem (Hindry–Silverman, as refined by Petsche)

Let K be a number field, let E/K be an elliptic curve, let $d = [K : \mathbb{Q}]$, and let $\sigma = \sigma_{E/K}$. There are explicit absolute constants $C_1, C_2, C_3 > 0$ such that

$$\#\{P \in E(K) \mid \hat{h}_E(P) \leq \frac{\log N_{K/\mathbb{Q}}(\mathcal{D}_{E/K})}{C_3 d\sigma^2}\} \leq C_1 d\sigma^2 \log(C_2 d\sigma^2).$$

In particular:

1. $\#E(K)_{\mathrm{tor}} \leq C_1 d\sigma^2 \log(C_2 d\sigma^2)$.
2. $\hat{h}_E(P) \geq \frac{1}{C_4 d^3\sigma^6 \log^2(C_2 d\sigma^2)} \log N_{K/\mathbb{Q}}\mathcal{D}_{E/K}$ *for all non-torsion* $P \in E(K)$ *(for an absolute constant $C_4 > 0$).*

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Average values of local and global canonical heights

Let K be a global field, and let $\hat{h} = \hat{h}_E : E(\bar{K}) \to \mathbb{R}_{\geq 0}$ be the Néron–Tate canonical height on E.

For $P \neq 0$, we can write $\hat{h}(P) = \sum_{v \in M_K} \frac{d_v}{d} \lambda_v(P)$, where $\lambda_v : E(\mathbb{C}_v) \backslash \{0\} \to \mathbb{R}$ is the normalized Néron local height function.

Given a set $Z = \{P_1, \ldots, P_N\}$ of distinct points of $E(K)$, set

$$\Lambda(Z) = \frac{1}{N^2} \sum_{i,j} \hat{h}(P_i - P_j)$$

and

$$\Lambda_v(Z) = \frac{1}{N^2} \sum_{i \neq j} \lambda_v(P_i - P_j).$$

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Strategy of the proof

Key idea: The expression $\Lambda(Z)$ can be bounded from **above** in terms of the $\hat{h}(P_i)$'s using the *parallelogram law*, and each $\Lambda_v(Z)$ can be bounded from **below** by a negative quantity tending to 0 as $N \to \infty$ using Fourier-style averaging arguments.

At a fixed archimedean place v_0 of K, use the pigeonhole principle to pass to a subset Z' of Z of positive density such that all points of Z' are close to each other in $E(\mathbb{C})$.

This makes $\Lambda_{v_0}(Z')$ **large**, and for $N \gg 0$ all other $\Lambda_v(Z')$ will be almost non-negative. If the $\hat{h}(P_i)$ for $P_i \in Z'$ are sufficiently small, we get a contradiction.

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Archimedean case

Let $\lambda(z) = \lambda_{v_0}(z)$ for some fixed archimedean place v_0 of K.

Lemma (Hindry–Silverman)

If $z = r_1 + r_2\tau \in \mathbb{C}\backslash\{0\}$ *and* $\max\{|r_1|, |r_2|\} \leq \frac{1}{24}$, *then*

$$\lambda(z) \geq \frac{1}{288} \max\{1, \log|j(\tau)|\}.$$

Theorem (Elkies)

$$\Lambda_{v_0}(z) \geq -\frac{\log N}{2N} - \frac{1}{12N}\log^+|j_E| - \frac{16}{5N}.$$

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Hint of the proof of Elkies' theorem

The key tools in the proof of Elkies' theorem are the **eigenfunction expansion**

$$\lambda(x-y) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n(x)\overline{f_n(y)}$$

together with smoothing properties of *convolution with the heat kernel*.

Non-Archimedean case

Let $v \in M_K^\circ$ be a non-Archimedean place of K.
Following Rumely, we write

$$\lambda_v(P - Q) = i_v(P, Q) + j_v(P, Q)$$

where i_v is a non-negative term coming from arithmetic intersection theory.

If E has good reduction, then $j_v = 0$ and thus $\lambda_v(P - Q) \geq 0$.

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Tate curves

Now suppose j_E is non-integral, so that E/K has a Tate uniformization $E(\mathbb{C}_v) \xrightarrow{\sim} \mathbb{C}_v^*/q^{\mathbb{Z}}$, where $q = q_E$ is the Tate parameter of E. Let r be the composition of the Tate isomorphism with the map $\mathbb{C}_v^*/q^{\mathbb{Z}} \to \mathbb{R}/\mathbb{Z}$ sending z to $(\log |z|_v)/(\log |q|_v)$.

Then

$$j_v(P, Q) = \frac{1}{2}\mathbb{B}_2(r(P - Q)) \log |j_E|_v ,$$

where

$$\mathbb{B}_2(t) = (t - [t])^2 - \frac{1}{2}(t - [t]) + \frac{1}{6}$$

is the periodic second Bernoulli polynomial.

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Fourier analysis of $\mathbb{B}_2(t)$

Lemma

Let $E/\mathbb{C}_v$ be a Tate curve, and let $\delta_v = \log|j_E| > 0$. Then

$$\Lambda_v(Z) \geq \left(\frac{1}{\delta_v^2} - \frac{1}{N}\right)\frac{1}{12}\delta_v.$$

The proof is based on *Parseval's formula* together with the Fourier expansion

$$\mathbb{B}_2(t) = \frac{1}{2\pi^2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{1}{m^2}e^{2\pi imt}.$$

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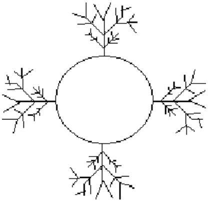
Interpretation via the Berkovich analytification

The map $r : E(\mathbb{C}_v) \to \mathbb{R}/\mathbb{Z}$ defined above can be identified with the retraction map to the skeleton Γ of the Berkovich analytic space $E_{\mathrm{Berk},v}$, which is isometric to a circle of length $\log |j_E|_v$:



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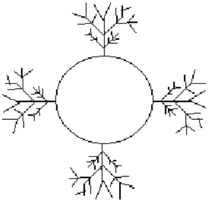

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Local discrepancy: Archimedean case

Let K be a number field. Given a sequence $Z = \{P_1, \ldots, P_N\}$ of distinct points in $E(\bar{K})$, we want to define a non-negative "smoothing" $\mathcal{D}_v(Z)$ of $\Lambda_v(Z)$, which we call the **local discrepancy** of Z.

Over $\mathbb{C}$, Elkies' theorem is proved by convolving with the heat kernel to get a 1-parameter family $\{\lambda_t\}_{t>0}$ of smooth functions $\lambda_t : E(\mathbb{C}) \to \mathbb{R}$ such that $\lim_{t\to 0} \lambda_t = \lambda_v$ and $\frac{1}{N^2}\sum_{i,j} \lambda_t(P_i - P_j) > 0$ for all t.

The best choice for our purposes is to take $t = 1/N$. We set, for Archimedean v:

$$\mathcal{D}_v(Z) := \frac{1}{N^2} \sum_{i,j} \lambda_{\frac{1}{N}}(P_i - P_j).$$

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In the non-Archimedean case, the singularity of λ_v at O comes from the non-negative intersection term $i_v(P, Q)$. This makes life easier than in the Archimedean case.

Here, a naive construction works just fine: we set $\lambda_v^*(O) = 0$ and $\lambda_v^*(P) = \lambda_v(P)$ for $P \neq O$, and define

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Although λ_v^* can be negative, Parseval's formula shows that $\mathcal{D}_v(Z) \geq 0$.

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The Height-Discrepancy Inequality

Define the **global discrepancy** of Z to be $\mathcal{D}(Z) := \sum_{v \in M_K} \frac{d_v}{d} \mathcal{D}_v(Z)$, and define $\hat{h}(Z) := \frac{1}{N} \sum_{P \in Z} \hat{h}(P)$.

Theorem (B.–Petsche)

Let K be a number field, let E/K be an elliptic curve, and let $Z = \{P_1, \ldots, P_N\}$ be a set of N distinct points in $E(\bar{K})$. Then

$$0 \le \mathcal{D}(Z) \le 4\hat{h}(Z) + \frac{1}{N}\left(\frac{1}{2}\log N + \frac{1}{12}h(j_E) + \frac{16}{5}\right).$$

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Local Discrepancy and Equidistribution

Intuitively, the local discrepancy measures how far $Z = \{P_1, \ldots, P_N\}$ is from being equidistributed with respect to the canonical measure at a given place v of K.

For $v \in M_K$ Archimedean, set $E_{\mathrm{Berk},v} := E(\mathbb{C})$ and let μ_v be the normalized Haar measure on $E_{\mathrm{Berk},v}$.

For $v \in M_K$ non-Archimedean, let μ_v be the normalized Haar measure on the circle Γ (pushed forward to $E_{\mathrm{Berk},v}$).

Theorem

Let Z_n be a sequence of finite subsets of $E(\mathbb{C}_v) \subset E_{\mathrm{Berk},v}$ with $\#Z_n \to \infty$. If $\mathcal{D}_v(Z_n) \to 0$, then the sequence Z_n is equidistributed with respect to μ_v.

Local Discrepancy and Equidistribution

Intuitively, the local discrepancy measures how far $Z = \{P_1, \ldots, P_N\}$ is from being equidistributed with respect to the canonical measure at a given place v of K.

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Global Discrepancy and Equidistribution of Small Points

Using the Height-Discrepancy Inequality, we obtain the following quantitative refinement of the Szpiro-Ullmo-Zhang equidistribution theorem for elliptic curves:

Theorem (B.–Petsche)

Let Z_n be a sequence of $\mathrm{Gal}(\bar{K}/K)$-invariant finite subsets of $E(\bar{K})$ with $\#Z_n \to \infty$. If there is a place $v \in M_K$ such that Z_n is not equidistributed in $E_{\mathrm{Berk},v}$ with respect to μ_v, then

$$\liminf \hat{h}(Z_n) \geq \frac{1}{4} \liminf \mathcal{D}(Z_n) > 0.$$

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Totally real and totally p-adic points

The following theorem is proved by combining the height-discrepancy inequality with Fourier analysis on $E(\mathbb{R})$ and a real-analytic version of the Tate uniformization:

Theorem (B.–Petsche)

Let $\mathbb{Q}_{\mathrm{tr}}$ be the maximal totally real subfield of $\bar{\mathbb{Q}}$, and let $E/\mathbb{Q}_{\mathrm{tr}}$ be an elliptic curve. There are explicit constants $C_1, C_2 > 0$ depending polynomially on $h(j_E)$ such that $\#E(\mathbb{Q}_{\mathrm{tr}})_{\mathrm{tor}} \leq C_1$ and $\hat{h}(P) \geq C_2$ for every non-torsion point $P \in E(\mathbb{Q}_{\mathrm{tr}})$.

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Elkies' theorem for Riemann surfaces

Elkies' result extends (with more or less the same proof) to compact Riemann surfaces X of arbitrary genus by using the Arakelov–Green function $g_\mu(x,y)$ attached to some volume form μ on $X(\mathbb{C})$ in place of $\lambda(x-y)$.

Definition: The Arakelov–Green function $g_\mu(x,y)$ is the unique function of two variables on $X(\mathbb{C})$ which is continuous away from the diagonal and satisfies:

- (Differential equation) $\Delta_y g_\mu(x,y) = \delta_x - \mu$.
- (Normalization) $\iint g_\mu(x,y)\mu(x)\mu(y) = 0$.

Theorem (Elkies)

$$\Lambda_\mu(Z) := \frac{1}{N^2}\sum_{i,j} g_\mu(P_i,P_j) \gg -\frac{\log N}{N}.$$

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A Non-Archimedean Elkies' inequality?

If v is non-Archimedean and X_{Berk} is the Berkovich analytification of an algebraic curve over $\mathbb{C}_v$, the space X_{Berk} deformation retracts onto a finite metrized graph Γ.

One can define the Arakelov–Green function $g_\mu(x, y)$ attached to any probability measure μ on X_{Berk} exactly as above, using Thuillier's Laplacian operator.

If μ is supported on Γ, it turns out (just as in the case of elliptic curves) that

$$g_\mu(x, y) = i(x, y) + j_\mu(x, y)$$

where $i(x, y)$ is a non-negative term coming from arithmetic intersection theory and $j_\mu(x, y)$ depends only on the retraction of x and y to Γ.

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An example

In fact, we have $j_\mu(x,y) = g_{\mu,\Gamma}(r(x), r(y))$ where $g_{\mu,\Gamma}(x,y)$ is a continuous function of two variables on Γ defined using the Laplacian operator on metrized graphs.

For example, when Γ is a circle, we have

$$g_{\mu,\Gamma}(x,y) = \frac{1}{2}\mathbb{B}_2(x-y)\ell(\Gamma).$$

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An Elkies-style inequality for metrized graphs

Theorem (B.–Rumely)

Let Γ be a metrized graph.

1. *The function $g_\mu(x,y)$ on Γ has an eigenfunction expansion*
$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n(x)\overline{f_n(y)}$$
which converges uniformly on $\Gamma \times \Gamma$ to $g_\mu(x,y)$.
2. *There exists $C > 0$ (depending on Γ and μ) such that*
$$\Lambda_\mu(Z) := \frac{1}{N^2} \sum_{i,j} g_\mu(P_i, P_j) \geq -\frac{C}{N}.$$

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Another example of an eigenfunction expansion on a metrized graph is the following:

Example: Let $\Gamma = [0,1]$ and let $\mu = \delta_0$. Then

$$g_\mu(x,y) = \min(x,y) = 8 \sum_{n \geq 1 \text{ odd}} \frac{\sin(\frac{n\pi x}{2})\sin(\frac{n\pi y}{2})}{\pi^2 n^2}.$$

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Canonical heights in arithmetic dynamics

Following Call and Silverman, one has an analogue of the Néron–Tate canonical height in arithmetic dynamics.
Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of degree $d \geq 2$ defined over a global field K.

We can lift ϕ to a map $F = (F_1(x,y), F_2(x,y)) : \mathbb{A}^2 \to \mathbb{A}^2$ where the F_i are homogeneous of degree d and have no common factor. Write $F^{(n)} = (F_1^{(n)}(x,y), F_2^{(n)}(x,y))$ for the nth iterate of F.

For $v \in M_K$, define the **local canonical height** $\hat{H}_{F,v} : \mathbb{C}_v^2 \backslash \{0\} \to \mathbb{R}$ by

$$\hat{H}_{F,v}(z_1,z_2) = \lim_{n\to\infty} \frac{1}{d^n} \log \max\{|F_1^{(n)}(z_1,z_2)|_v, |F_2^{(n)}(z_1,z_2)|_v\}.$$

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Canonical heights in arithmetic dynamics (continued)

The **global canonical height** of $z = (z_1 : z_2) \in \mathbb{P}^1(\bar{K})$ is defined to be

$$\hat{h}_\phi(z) = \sum_{v \in M_K} \frac{d_v}{d} \hat{H}_{F,v}(z_1, z_2).$$

By the **product formula**, this is independent of the lift F and of the coordinate representation $(z_1 : z_2)$.

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Canonical measures in arithmetic dynamics

For each place v of K there is a **canonical measure** $\mu_{\phi,v}$ on $\mathbb{P}^1_{\mathrm{Berk},v}$ which governs equidistribution of periodic points and iterated preimages. In the Archimedean case, the canonical measure is the well-known *measure of maximal entropy* studied by Brolin, Lyubich, and Freire-Lopes-Mañé.

For example, if $\phi(z) = z^2$ then for v Archimedean $\mu_{\phi,v}$ is Haar measure on the complex unit circle in $\mathbb{P}^1(\mathbb{C})$.

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Arakelov-Green functions in arithmetic dynamics

There is a corresponding Arakelov-Green function at each place.

The Arakelov-Green function for $\phi(z) = z^2$ is given in both the Archimedean and non-Archimedean cases by the formula

$$g_{\phi,v}((x_1, y_1), (x_2, y_2)) \\ = -\log|x_1 y_2 - x_2 y_1|_v + \log\max\{|x_1|_v, |y_1|_v\} + \log\max\{|x_2|_v, |y_2|_v\}.$$

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An explicit formula for the dynamical Arakelov-Green function

The formula for $g_{\phi,v}$ when $\phi(z) = z^2$ generalizes nicely to arbitrary rational maps:

Theorem (B.–Rumely)

Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$. For any place v of K, we have

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Relation with the dynamical canonical height

Corollary: For every $x, y \in \mathbb{P}^1(\bar{K})$ we have

$$\hat{h}_\phi(x) + \hat{h}_\phi(y) = \sum_{v \in M_K} \frac{d_v}{d} g_{\varphi,v}(x, y).$$

A Mahler-Elkies style lower bound for average values of the dynamical Arakelov-Green function

Theorem (B.)

Let $\phi \in \mathbb{C}_v(z)$ be a rational map of degree $d \geq 2$. There exists a constant $C > 0$ depending on ϕ such that if $Z = \{P_1, \ldots, P_N\}$ is a set of N distinct points in $\mathbb{P}^1(\bar{K})$, then

$$\Lambda_{\phi,v}(Z) := \frac{1}{N^2} \sum_{i,j} g_{\phi,v}(P_i, P_j) \geq -\frac{C \log N}{N}.$$

Commentary

Note that we don't have Fourier analysis or eigenfunction expansions available to us in this setting.

The proof uses the explicit formula for $g_{\phi,v}(x,y)$ together with a rather elaborate algebraic analysis of certain determinants and resultants.

It generalizes an old estimate due to Mahler for the usual Weil height based on van der Monde determinants and Hadamard's inequality.

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A global application

We deduce the following Hindry–Silverman type estimate:

Theorem (B.)

There are constants $A, B > 0$ depending on ϕ and L such that if $[L : K] = D$ then

$$\#\{P \in \mathbb{P}^1(L) \mid \hat{h}_\phi(P) \le \frac{A}{D}\} \le B \cdot D \log D.$$

The proof uses a pigeonhole principle argument at a fixed place of K (using the compactness of $\mathbb{P}^1(\mathbb{C})$ or $\mathbb{P}^1_{\mathrm{Berk},v}$) together with the Mahler-Elkies style lower bound from the previous slide.

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A lower bound for the degree of the field of definition of preperiodic points

Corollary: There is a constant $C > 0$ depending on ϕ and L such that if $P_1, \ldots, P_N$ are distinct preperiodic points of ϕ defined over L, then $[L : K] \geq C \frac{N}{\log N}$.

Note that the corollary is nearly sharp in the case $\phi(z) = z^2$, since $[\mathbb{Q}(\zeta_N) : \mathbb{Q}] = \varphi(N) \gg \frac{N}{\log \log N}$.

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An open problem

It is an interesting open problem to study the dependence of the constants A, B on the map ϕ in the estimate:

$$\#\{P \in \mathbb{P}^1(L) \mid \hat{h}_\phi(P) \leq \frac{A}{D}\} \leq BD \log D.$$

The constants are ineffective as it stands because of the compactness argument invoked. It would be very interesting to have an analogue in arithmetic dynamics of the Hindry-Silverman theorem (that Szpiro's conjecture implies uniform boundedness of rational torsion points) relating some variant of the ABC Conjecture to the Morton-Silverman conjecture on uniform boundedness of rational preperiodic points.

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Potentially good reduction and isotriviality

Let v be non-Archimedean. We say that $\phi \in \mathbb{C}_v(T)$ has **potentially good reduction** if it has good reduction after a change of coordinates (i.e., after conjugating by a Möbius transformation).

We say that ϕ has **genuinely bad reduction** if it does not have potentially good reduction.

If K is a function field (with arbitrary constant field), we say that $\phi \in K(T)$ is **isotrivial** if, after a change of coordinates and a finite extension of K, it is defined over the field of constants.

Theorem (B.)

Let K be a function field, and let $\phi \in K(T)$ be a rational map of degree at least 2. Then ϕ is isotrivial if and only if ϕ has potentially good reduction over $\mathbb{C}_v$ for all $v \in M_K$.

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Potentially good reduction and dynamical Green functions

Theorem

ϕ has genuinely bad reduction over $\mathbb{C}_v$ if and only if $g_{\phi,v}(x,x) > 0$ for all $x \in \mathbb{P}^1_{\mathrm{Berk},v} \backslash \mathbb{P}^1(\mathbb{C}_v)$.

Corollary: If ϕ has genuinely bad reduction, then there exists a constant $\beta > 0$ and a covering of $\mathbb{P}^1(\mathbb{C}_v)$ by finitely many analytic open sets $V_1, \ldots, V_t$ such that for each $1 \le i \le t$, we have $g_{\phi,v}(x,y) \ge \beta$ for all $x, y \in V_i$.

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Isotriviality and preperiodicity

Using the above results and a Hindry-Silverman style pigeonhole argument, we deduce the following global consequence:

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It would be interesting to find good explicit bounds for t and β in terms of the map ϕ. This would yield an extension of Benedetto's "$s \log s$" bound for the number of preperiodic points of a polynomial map ϕ to arbitrary rational maps.

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Happy Birthday, Joe!