Does ω^* know its right hand from its left?

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PALS

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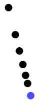

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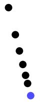

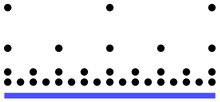

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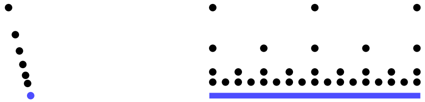

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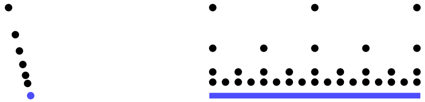

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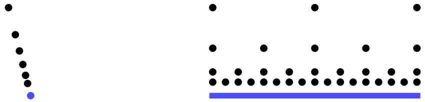

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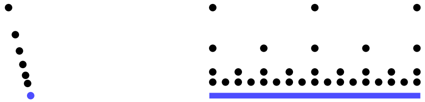

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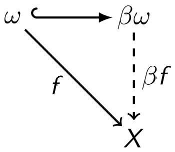
$\beta\omega$ is the *largest* compactification of ω:
i.e., if $\gamma\omega$ is any other compactification of ω, then there is a continuous surjection $\pi : \beta\omega \to \gamma\omega$ that fixes ω.

The space $\beta\omega$

$\beta\omega$ is the unique compactification of ω with the following property (called the *Stone extension property*):

- every function mapping ω into a compact Hausdorff space extends continuously to $\beta\omega$.



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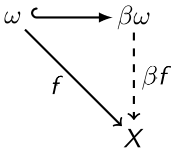

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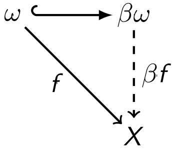

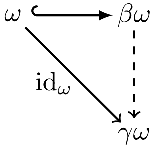

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The fact that $\beta\omega$ is the largest compactification of ω follows from the extension property.

The space ω^*

The space of all non-principal ultrafilters on ω, known as the *Stone-Čech remainder* of ω, is denoted

$$\omega^* = \beta\omega \setminus \omega.$$

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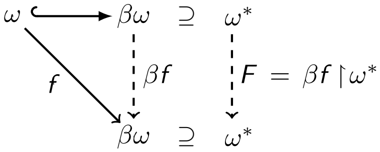

Explicitly, for any ultrafilter $u \in \omega^*$, $F(u) = \{f[A] : A \in u\}$.

Trivial self-homeomorphisms

For example, the *shift map* $\sigma : \omega^* \to \omega^*$ is defined by setting

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Theorem (W. Rudin, 1956)

The Continuum Hypothesis (CH) implies there are $2^{\mathfrak{c}}$ self-homeomorphisms of ω^*. In particular, some of them are non-trivial.

A theorem of Shelah

In contrast with Rudin's theorem, we have the following.

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- (Vecličković, 1992) OCA+MA implies that all self-homeomorphisms of ω^* are trivial.
- (Farah, 2000) OCA+MA imposes strong restrictions on all self-maps of ω^* (not just self-homeomorphisms), and there is a sense in which all of them are nearly trivial.

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Two dynamical systems (X, f) and (Y, g) are *conjugate* if there is a homeomorphism $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$.

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$$\begin{CD} X @>f>> X \\ @V\phi VV @VV\phi V \\ Y @>>g> Y \end{CD}$$

This is the natural notion of isomorphism in the category of dynamical systems. A weaker notion is that of a *factor map*, which is defined the same way, except that ϕ is only required to be a continuous surjection.

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Corollary (van Douwen and Shelah)

It is consistent that (ω^*, σ) and (ω^*, σ^{-1}) are not conjugate.

In fact, it is consistent that there is not even a factor map from (ω^*, σ) to (ω^*, σ^{-1}) or vice versa.

The main theorem

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In what time remains, we will sketch a part of the proof.

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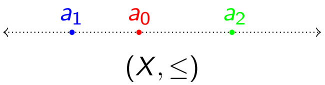

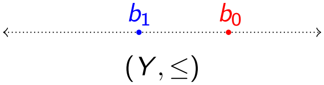

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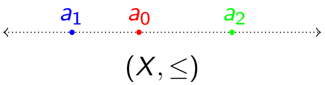

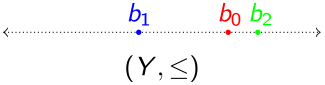

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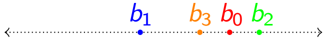

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The rest is a routine construction by recursion.

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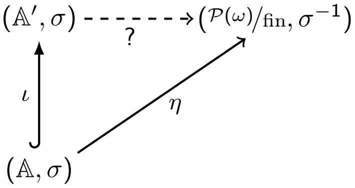

The second item asks for a kind of "lifting property" for $(\mathcal{P}(\omega)/\mathrm{fin}, \sigma)$: given two embeddings as shown, is there an embedding of $(\mathbb{A}', \sigma)$ that completes the diagram?

Not so fast . . .

A very annoying fact:

The second bullet point on the previous slide is not generally true.

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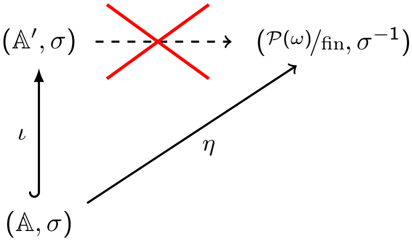
The second bullet point on the previous slide is not generally true. More precisely, there are countable substructures $(\mathbb{A}, \sigma^{-1})$ *and* $(\mathbb{A}', \sigma^{-1})$ *of* $(\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$*, with* $\mathbb{A} \subseteq \mathbb{A}'$*, and an embedding* $\eta : (\mathbb{A}, \sigma^{-1}) \to (\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$*, such that there is no embedding* $\bar{\eta} : (\mathbb{A}', \sigma^{-1}) \to (\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$ *with* $\bar{\eta} \circ \iota = \eta$*.*



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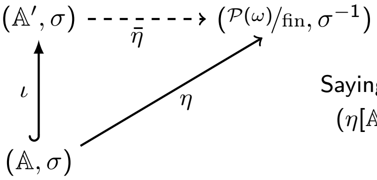
Main Lemma:

Suppose $(\mathbb{A}, \sigma^{-1})$ *and* $(\mathbb{A}', \sigma^{-1})$ *are countable substructures of* $(\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$ *with* $\mathbb{A} \subseteq \mathbb{A}'$*, and* $\eta : (\mathbb{A}, \sigma) \to (\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$ *is an "elementary" embedding. Then* η *extends to an embedding* $\bar{\eta} : (\mathbb{A}', \sigma^{-1}) \to (\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$*, so that* $\bar{\eta} \circ \iota = \eta$*.*



Saying η is "elementary" means
$(\eta[\mathbb{A}], \sigma^{-1}) \prec (\mathcal{P}(\omega)/\mathrm{fin}, \sigma^{-1})$

A better back-and-forth

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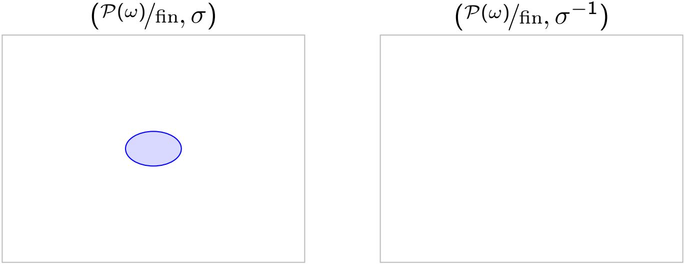

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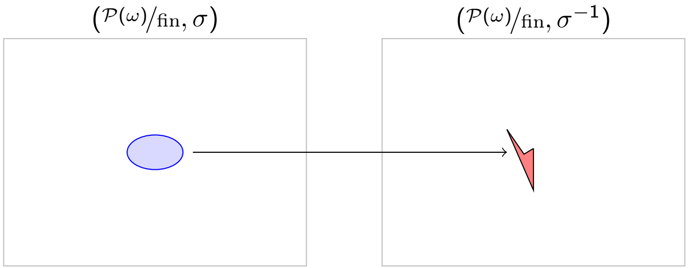

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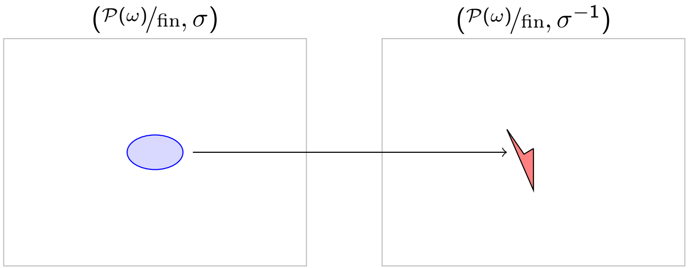

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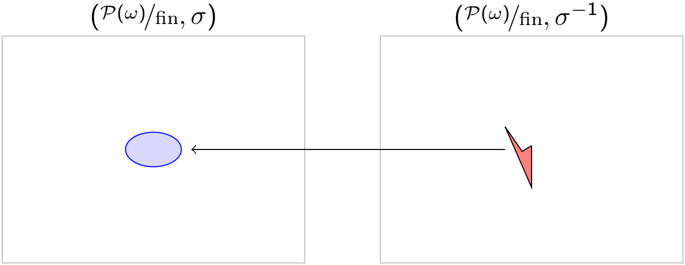

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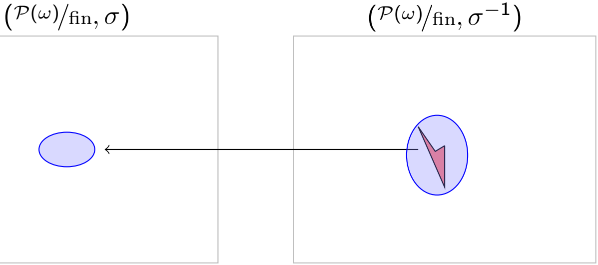

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(6) Find a countable elementary substructure of $(\mathcal{P}(\omega)/\text{fin}, \sigma^{-1})$ that contains the image of our embedding.

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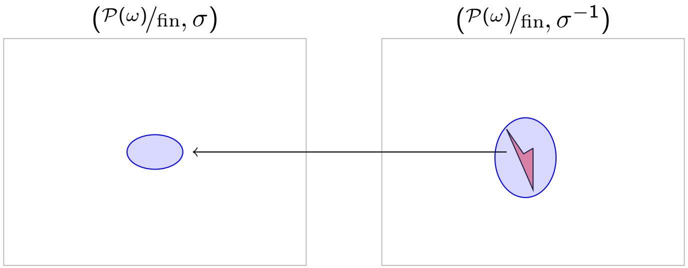

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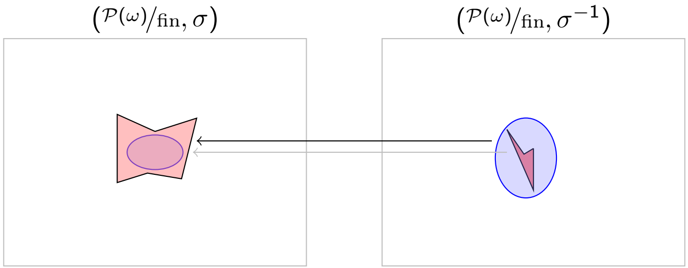

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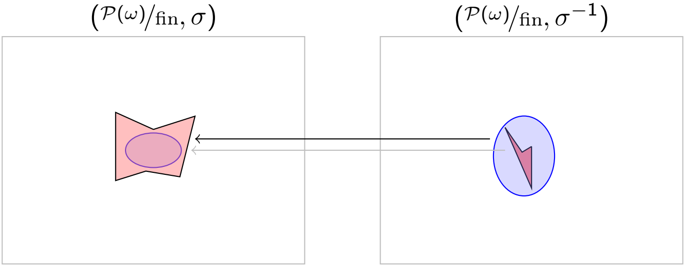

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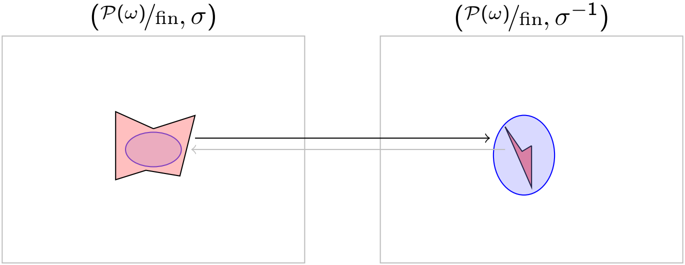

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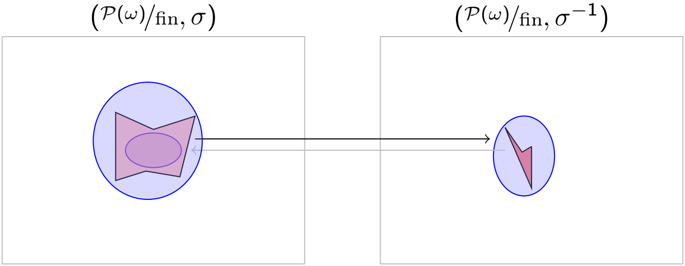

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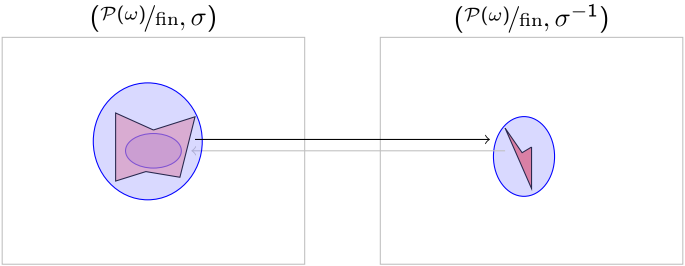

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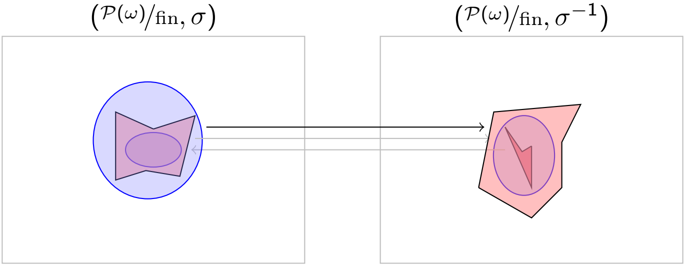

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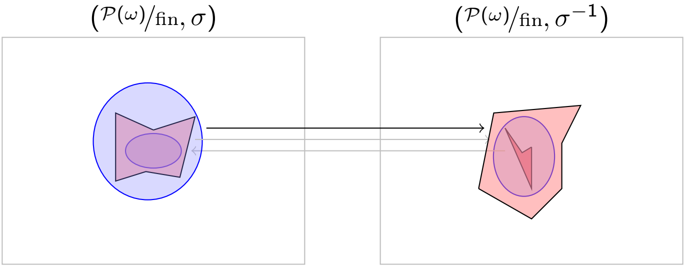

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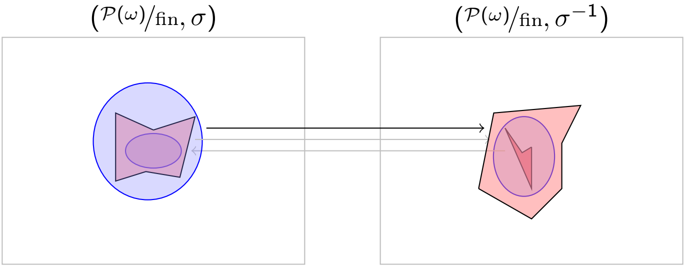

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(12) At stage α, be sure that the elementary structure used on each side contains the α^{th} member of $\mathcal{P}(\omega)/\mathrm{fin}$.

Two corollaries and a question

Observe that, when we take unions at the countable limit stages of the construction, we get elementary substructures on both sides, and an isomorphism between them.

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Question

Is there an order-reversing self-homeomorphism of $[0, \infty)^*$?

The end

Thank you for listening