Minion homomorphisms and valued CSP

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March 11th 2021

Valued CSP

- Given $q \in \mathbb{Q}$. Is there a $\sigma\colon V \to D$ so that

$$r_1 \cdot R_1(\sigma(v_{11}), \sigma(v_{12}), \dots) + \dots + r_k \cdot R_k(\sigma(v_{k1}), \dots) \leq q?$$

- Here $R_i\colon D^n \to \mathbb{Q} \cup \{\infty\}$
- New: Parameters $r_i \in \mathbb{Q} \cap [0, \infty)$ are part of the input
- Non-uniform PVCSP: Fixed D and the possible cost functions R

Weighted polymorphisms

- $F\colon A^n \to B$ is weighted sum of operations
- Notation for this seminar:

$$F_I(\pi_1)\boldsymbol{\pi}_1 + F_I(\pi_2)\boldsymbol{\pi}_2 + \dots + F_I(\pi_n)\boldsymbol{\pi}_n \to F_O(g)\boldsymbol{g} + F_O(h)\boldsymbol{h} + \dots .$$

- Today's view of submodularity

$$1 \cdot \boldsymbol{\pi}_1 + 1 \cdot \boldsymbol{\pi}_2 \to 1 \cdot \wedge + 1 \cdot \vee$$

Weighted polymorphisms

- Let $\mathbb{A}, \mathbb{B}$ be weighted relational structures
- For each n, the set $\mathrm{wPol}_n(\mathbb{A}, \mathbb{B})$ consists of (F_I, F_O) such that for any k-ary R and any $\overline{c}_1, \dots, \overline{c}_n \in A^k$ we have

$$F_I(\pi_1)R^{\mathbb{A}}(\overline{c}_1) + \cdots + F_I(\pi_n)R^{\mathbb{A}}(\overline{c}_n) \geq \sum_h F_O(h)R^{\mathbb{B}}(h(\overline{c}_1, \dots, \overline{c}_n)).$$

- Submodularity again:

$$R^{\mathbb{A}}(\overline{c}_1) + R^{\mathbb{A}}(\overline{c}_2) \geq R^{\mathbb{B}}(\overline{c}_1 \wedge \overline{c}_2) + R^{\mathbb{B}}(\overline{c}_1 \vee \overline{c}_2)$$

Homomorphisms ought to give gadget reductions

- Barto, Bulín, Opršal, Krokhin (BBOK), 2019: Homomorphisms of minions give gadget reductions
- Can we do this for PVCSP with weighted polymorphisms?
- What is even the right notion of a homomorphism for weighted polymorphism clones?

Support minion

$$F_I(\pi_1)\boldsymbol{\pi}_1 + F_I(\pi_2)\boldsymbol{\pi}_2 + \cdots + F_I(\pi_n)\boldsymbol{\pi}_n \to F_O(g)\boldsymbol{g} + F_O(h)\boldsymbol{h} + \dots.$$

- Denote by $\mathrm{wPol}_n^+(\mathbb{A},\mathbb{B})$ all f such that there is some $F \in \mathrm{Pol}_n(\mathbb{A},\mathbb{B})$ with $F_O(f) > 0$
- Warning: In general $\mathrm{wPol}_n^+(\mathbb{A},\mathbb{B}) \subsetneq \mathrm{Pol}(\mathbb{A},\mathbb{B})$
- $\mathrm{Pol}^+(\mathbb{A},\mathbb{B}) = \bigcup_{n=1}^{\infty} \mathrm{Pol}_n^+(\mathbb{A},\mathbb{B})$ is nonempty and closed under minor-taking
- The minor of f via σ, is defined as

$$f^\sigma(x_1,\dots,x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

- Minion homomorphism $\phi\colon \mathrm{Pol}_n^+(\mathbb{A},\mathbb{B}) \to \mathrm{Pol}_n^+(\mathbb{C},\mathbb{D})$ commutes with minor-taking: $\phi(f^\sigma) = \phi(f)^\sigma$

Homomorphisms of weighted polymorphisms (simplified)

- Weighted minion homomorphism $\phi\colon \mathrm{Pol}(\mathbb{A},\mathbb{B}) \to \mathrm{Pol}(\mathbb{C},\mathbb{D})$ is a minion homomorphism such that when

$$\sum_{i=1}^{n} F_I(\pi_i)\boldsymbol{\pi}_i \to \sum_{f\in \mathrm{Pol}_n^+(\mathbb{A},\mathbb{B})} F_O(f)\boldsymbol{f}$$

  is from $\mathrm{wPol}_n(\mathbb{A},\mathbb{B})$, then

$$\sum_{i=1}^{n} F_I(\pi_i)\boldsymbol{\pi}_i \to \sum_{f\in \mathrm{Pol}_n^+(\mathbb{A},\mathbb{B})} F_O(f)\phi(f).$$

  lies in $\mathrm{wPol}_n(\mathbb{C},\mathbb{D})$
- The not-simplified version is a probability distribution over ϕ's

Valued Promise Minor Condition Problem ($\mathrm{PVMC}_N(\mathbb{A},\mathbb{B})$):

INPUT: $q \in \mathbb{Q}$; a finite set of minor conditions Σ with operation symbols $f_1, \dots, f_n$ and rational valued maps $(\alpha_i, \beta_i)_{i=1}^n$

Arities of f_i's at most N

Each (α_i, β_i) is compatible with $\mathrm{wPol}(\mathbb{A},\mathbb{B})$

OUTPUT "Yes": Exists an assignment $\xi\colon [n] \to [N]$ such that $f_i \mapsto \pi_{\xi(i)}$ satisfies Σ and we have $\sum_{i=1}^n \alpha_i(\xi(i)) \leq q$.

OUTPUT "No": No arity-respecting assignment $\xi\colon \{f_1, \dots, f_n\} \to \mathrm{wPol}^+(\mathbb{A},\mathbb{B})$ satisfying Σ such that $\sum_{i=1}^n \beta_i(\xi(f_i)) \leq q$

Compatibility: (α, β) is compatible with $\mathrm{wPol}(\mathbb{A},\mathbb{B})$ if for any $F \in \mathrm{wPol}_n(\mathbb{A},\mathbb{B})$

$$\sum_{i=1}^{n} F_I(\pi_i)\alpha(i) \geq \sum_{h} F_O(h)\beta(h)$$

Example

- Pick $\mathbb{A}, \mathbb{B}$ on $\{0, 1\}$
- Let $n = 2$, both ops binary
- Let Σ be the system

$$f(x, y) \approx g(y, x)$$
$$f(y, x) \approx g(y, x)$$

- Suppose (α_i, β_i) are the same for $i = 1, 2$
- $\alpha(\pi_1) = \alpha(\pi_2) = 1$ and $\beta(f) = f(0, 1) + f(1, 0)$ with $q = 1$ for $i = 1, 2$
- There is no projection such that $f(x, y) \approx f(y, x)$, so this is not a "Yes" instance
- Is this a "No" instance? It depends

Example, cont.

- Not a “No” instance if there exist $f, g \in \mathrm{wPol}_2^+(\mathbb{A}, \mathbb{B})$ such that

$$f(x,y) \approx g(y,x)$$
$$f(y,x) \approx g(y,x)$$

  and $f(0,1) + f(1,0) + g(0,1) + g(1,0) \leq 1$
- Really we just want a commutative f such that $4f(0,1) \leq 1$
- Thus $f(0,1) = f(1,0) = 0$ and f is one of $\wedge$, NOR, XNOR or constant 0

Example, that pesky compatibility

- In order for the example to be a valid instance, we need (α_i, β_i) to be compatible with $\mathrm{wPol}_2(\mathbb{A}, \mathbb{B})$
- What does that mean?
- If $F \in \mathrm{wPol}_2(\mathbb{A}, \mathbb{B})$ then

$$F_I(\pi_1)\alpha(\pi_1) + F_I(\pi_2)\alpha(\pi_2) \geq \sum_h F_O(h)\beta(h)$$

- That simplifies to

$$F_I(\pi_1) + F_I(\pi_2) \geq \sum_h F_O(h)(h(0,1) + h(1,0))$$

- Equivalent to $F \in \mathrm{wPol}((\{0,1\}, R^{\mathbb{A}}), (\{0,1\}, R^{\mathbb{B}}))$ where

$$R^{\mathbb{A}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = R^{\mathbb{A}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 1, \quad R^{\mathbb{A}} = \infty \text{ else}, \quad R^{\mathbb{B}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = b + c$$

Reductions of PVCSP through PVMC

- Given $\phi\colon \operatorname{wPol}(\mathbb{A},\mathbb{B}) \to \operatorname{wPol}(\mathbb{C},\mathbb{D})$
- Main idea the same as in the BBOK paper

$$\mathrm{PVCSP}(\mathbb{C},\mathbb{D}) \to \mathrm{PVMC}_N(\mathbb{C},\mathbb{D}) \to \mathrm{PVMC}_N(\mathbb{A},\mathbb{B}) \to \mathrm{PVCSP}(\mathbb{A},\mathbb{B})$$

- Trivial reduction: $\mathrm{PVMC}_N(\mathbb{C},\mathbb{D}) \to \mathrm{PVMC}_N(\mathbb{A},\mathbb{B})$ by doing nothing
- What remains: Equivalence of $\mathrm{PVCSP}(\mathbb{A},\mathbb{B})$ and $\mathrm{PVMC}_N(\mathbb{A},\mathbb{B})$ for suitably big N

$\mathrm{PVMC}_N(\mathbb{C}, \mathbb{D}) \to \mathrm{PVMC}_N(\mathbb{A}, \mathbb{B})$ by doing nothing

- Given $\phi\colon \mathrm{wPol}(\mathbb{A}, \mathbb{B}) \to \mathrm{wPol}(\mathbb{C}, \mathbb{D})$
- "Yes" instances go to "Yes" instances always
- "Yes" instance: $\exists \xi\colon [n] \to [N]$ such that $f_i \mapsto \pi_{\xi(i)}$ satisfies Σ and we have $\sum_{i=1}^n \alpha_i(\xi(i)) \leq q$.
- "No" instances of $\mathrm{PVMC}_N(\mathbb{C}, \mathbb{D})$ go to "No" instances of $\mathrm{PVMC}_N(\mathbb{A}, \mathbb{B})$ thanks to ϕ
- If $\exists \xi\colon \{f_1, \dots, f_n\} \to \mathrm{wPol}^+(\mathbb{A}, \mathbb{B})$ satisfying Σ such that $\sum_{i=1}^n \beta_i(\xi(f_i)) \leq q$, then $\phi \circ \xi$ works for $\mathrm{PVMC}_N(\mathbb{C}, \mathbb{D})$

Reduction from PVCSP($\mathbb{C}, \mathbb{D}$) to PVMC$_N$($\mathbb{C}, \mathbb{D}$)

- Idea from BBOK
- Example with $C = \{0, 1\}$

$$R\begin{pmatrix} x \\ y \end{pmatrix} + S\left(y\right) \leq 42$$

- Enumerate support sets of relations of $\mathbb{C}$
- $R^{\mathbb{C}} < \infty$ for

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

  and $S^{\mathbb{C}}(0) < \infty$
- Then Σ becomes like in BBOK

$$f_x(x_0, x_1) \approx f_R(x_1, x_1, x_0)$$
$$f_y(x_0, x_1) \approx f_R(x_1, x_0, x_1)$$
$$f_y(x_0, x_1) \approx f_S(x_0)$$

Reduction from PVCSP($\mathbb{C}, \mathbb{D}$) to PVMC$_N$($\mathbb{C}, \mathbb{D}$), cont.

- $R^{\mathbb{C}} < \infty$ for
$$\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$$
and $S^{\mathbb{C}}(0) < \infty$
- $\alpha_R(i)$ cost of $R^{\mathbb{C}}$ of i-th tuple
- $\beta_R(h) = R^{\mathbb{D}}\left(h\begin{pmatrix}1 & 1 & 0\\1 & 0 & 1\end{pmatrix}\right)$
- (α_S, β_S) similarly
- (α_x, β_x), (α_y, β_y) zero everywhere
- Exercise: This is compatible with wPol($\mathbb{C}, \mathbb{D}$)
- Cheap solution of PVCSP($\mathbb{C}, \mathbb{D}$) in $\mathbb{C} \to$ a cheap solution of PVMC$_N$($\mathbb{C}, \mathbb{D}$) in projections
- No cheap solution of PVCSP($\mathbb{C}, \mathbb{D}$) in $\mathbb{D} \to$ no cheap solution of PVMC$_N$($\mathbb{C}, \mathbb{D}$) in Pol$^+$($\mathbb{C}, \mathbb{D}$)

Reduction from $\mathrm{PVMC}_N(\mathbb{A}, \mathbb{B})$ to $\mathrm{PVCSP}(\mathbb{A}, \mathbb{B})$

- Works for any N
- Given (α_i, β_i) compatible with $\mathrm{wPol}_i(\mathbb{A}, \mathbb{B})$
- Need to show that we can simulate (α_i, β_i) using a PVCSP instance
- Needs: We can emulate each (α_i, β_i) by a pair of relations
- Farkas' lemma is handy here
- Then satisfying solution $\Rightarrow$ PVMC has solution in projections
- No cheap map to $\mathbb{B}$ $\Rightarrow$ no cheap solution in $\mathrm{Pol}^+(\mathbb{A}, \mathbb{B})$

Example

- Take the PVMC example from before
- Since (α, β) are compatible with $(\mathbb{A}, \mathbb{B})$, Farkas' lemma gives us, for example, that

$$\alpha(\pi_1) \geq 3R^{\mathbb{A}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 7R^{\mathbb{A}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\alpha(\pi_2) \geq 3R^{\mathbb{A}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 7R^{\mathbb{A}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta(h) \leq 3R^{\mathbb{B}} \left(h \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right) + 7R^{\mathbb{B}} \left(h \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \right)$$

Example, cont.

- Consider the PVCSP instance

$$3 \cdot R(x_{10}, x_{01}, x_{00}) + 7R(x_{10}, x_{10}, x_{11}) \leq 1/2$$

- Under the rug: A constraint that ensures that $\mathbb{B}$ we always have have $x_{ij} = h(i,j)$ for some $h \in \mathrm{wPol}_2^+(\mathbb{A}, \mathbb{B})$
- Cheap solution of $\mathrm{PVMC}_N(\mathbb{A}, \mathbb{B})$ in projections $\Rightarrow$ a cheap solution of $\mathrm{PVCSP}(\mathbb{A}, \mathbb{B})$ in $\mathbb{A}$
- No cheap solution of $\mathrm{PVMC}_N(\mathbb{A}, \mathbb{B})$ in $\mathrm{Pol}^+(\mathbb{A}, \mathbb{B})$ $\Rightarrow$ no cheap solution of $\mathrm{PVCSP}(\mathbb{A}, \mathbb{B})$ in $\mathbb{B}$
- Also under the rug: Representing equation system

$$f(x,y) \approx g(y,x)$$
$$f(y,x) \approx g(y,x)$$

by gluing together some x_{01} and x_{10}.