The correct category for valued CSP

Alexandr Kazda

CU Boulder

February 4th 2021

Valued CSP

- Given $q \in \mathbb{Q}$. Is there a $\sigma\colon V \to D$ so that

$$R_1(\sigma(v_{11}), \sigma(v_{12}), \dots) + R_2(\sigma(v_{21}), \dots) + \dots + R_k(\sigma(v_{k1}), \dots) \leq q?$$

- Here $R_i\colon D^n \to \mathbb{Q} \cup \{\infty\}$
- Non-uniform VCSP: Fixed D and the possible cost functions R_i

VCSP vs. CSP

- If we only choose $R_i \colon : D^n \to \nvdash \cup \{\infty\}$, we get CSP
- Inequality

$$R_1(\sigma(v_{11}), \sigma(v_{12}), \dots) + R_2(\sigma(v_{21}), \dots) + \dots + R_k(\sigma(v_{k1}), \dots) \leq 0$$

becomes

$$\bigvee_{i=1}^{k} (\sigma(v_{i1}), \sigma(v_{i2}), \dots) \in \operatorname{dom} R_i$$

Isn't this done?

- Complexity dichotomy (in P vs. NP-hard) proved by Kolmogorov, Krokhin, Rolínek in 2015
- But we can make the math more beautiful. . .

Weighted polymorphisms

- $\mathbb{A} = (A; R)$ a valued relational structure
- $\psi \colon A^n \to A$ is weighted sum of operations
- Compatibility condition for $n = 2$: For all $\overline{x}_1, \overline{x}_2$ in A^2

$$\alpha R(\overline{x}_1) + (1-\alpha) R(\overline{x}_2) \geq \sum_{h \colon A^2 \to A} \psi(h) R(h(\overline{x}_1, \overline{x}_2))$$

- Example: Submodularity

$$1/2 \cdot R(\overline{x}) + 1/2 \cdot R(\overline{y}) \geq 1/2 \cdot (R(\overline{x} \wedge \overline{y})) + 1/2 \cdot (R(\overline{x} \vee \overline{y}))$$

- Kolmogorov, Krokhin, Rolínek: If $\mathbb{A}$ is a rigid core then VCSP($\mathbb{A}$) is in P if $\mathbb{A}$ has a cyclic fractional polymorphism; VCSP($\mathbb{A}$) is NP-hard else.

Category theory

- In (P)CSP the symmetries were given by all morphisms $\mathbb{A}^n \to \mathbb{B}$
- We can define $\mathsf{Pol}(\mathbb{A}, \mathbb{B}) = \mathsf{Hom}(\mathbb{A}^n, \mathbb{B})$ in any category with powers
- This does not ensure CSP-style reductions, but we can hope...
- If we have products, we can try to construct $\mathbb{B}^{\mathbb{C}}$ as right adjoint functor to $\times$:

$$\mathsf{Hom}(\mathbb{A} \times \mathbb{C}, \mathbb{B}) \simeq \mathsf{Hom}(\mathbb{A}, \mathbb{B}^{\mathbb{C}})$$

- In Set, B^C is just the set of all mappings $C \to B$; the isomorphism goes by $f \mapsto (a \mapsto (c \mapsto f(a, c)))$
- $\mathbb{B}^{\mathbb{C}}$ is useful, e.g. in graph theory

Naive VCSP category

- Why not just take all valued relational structures of a given signature?
- Morphisms, say mappings that do not increase "cost"
- Eg. $\mathbb{A} = (A; R^{\mathbb{A}})$ where $R^{\mathbb{A}} \colon A^2 \to \mathbb{Q} \cup \{\infty\}$?
- Say $h \colon A \to B$ s.t. for all $\overline{a}$

$$R^{\mathbb{A}}(\overline{a}) \geq R^{\mathbb{B}}(h(\overline{a}))$$

- Products are problematic...

Defective products

- Natural candidate for $\mathbb{A}^2$ has universe A^2, but what about $R^{\mathbb{A}^2}$?
- We want projections to be homomorphisms $\mathbb{A}^2 \to \mathbb{A}$, so $\forall a_{ij}$

$$R^{\mathbb{A}^2}((a_{11}, a_{12}), (a_{21}, a_{22})) \geq R^{\mathbb{A}}(a_{11}, a_{21}), R^{\mathbb{A}}(a_{12}, a_{22})$$

- Maybe

$$R^{\mathbb{A}^2}((a_{11}, a_{12}), (a_{21}, a_{22})) = \max(R^{\mathbb{A}}(a_{11}, a_{21}), R^{\mathbb{A}}(a_{12}, a_{22}))?$$

- We get projections, but $\mathbb{A}^2 \to \mathbb{A}$ is not $\mathrm{Pol}_2(\mathbb{A})$
- Moreover: VCSP is not a homomorphism problem here

More correct VCSP category

- Objects: Convex combinations of valued relational structures
- Morphisms: *Distributions* over mappings $\phi : \mathfrak{A} \to \mathfrak{B}$ such that for each structure $\mathbb{B} \in \mathfrak{B}$ there is a relational structure $\mathbb{A} \in \mathfrak{A}$ such that "on average" $\phi \colon \mathbb{A} \to \mathbb{B}$ is a homomorphism

$$R^{\mathbb{A}}(\overline{x}) \geq \sum_{h\colon A \to B} \phi(h) R^{\mathbb{B}}(h(\overline{x}))$$

- $\mathrm{Hom}(\{\mathbb{A}\}^n, \{\mathbb{A}\})$ are weighted polymorphisms

Pesky ∞

- We want to use convex geometry and ∞ gets in the way
- Instead of $R^{\mathbb{A}}(\overline{a}) = \infty$, make it so that there are $\mathbb{A} \in \mathfrak{A}$ with $R^{\mathbb{A}}(\overline{a})$ arbitrarily big
- Example: Instead of

$$R(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{else,} \end{cases}$$

take $\mathfrak{A} = \{(A; R^{(s)})\colon s \in \mathbb{Q} \cap [0,\infty)\}$ where

$$R^{(s)}(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ s & \text{else} \end{cases}$$

- Now mappings $\mathfrak{A} \to \mathfrak{B}$ don't have to worry about (x,x) ...
- ... and mappings $\mathfrak{B} \to \mathfrak{A}$ can't send finite cost pairs to (x,x)

CSP becomes a homomorphism problem

- Let $\mathfrak{B} = \{\mathbb{B}\}$
- Suppose $\mathfrak{A}$ contains all structures on A with $R^{\mathbb{A}}$ such that

$$\sum_{\overline{a}\in X} R^{\mathbb{A}}(\overline{a}) \leq q$$

- Then $\phi\colon \mathfrak{A} \to \mathfrak{B}$ iff

$$R^{\mathbb{A}}(\overline{a}) \geq R^{\mathbb{B}}(\phi(\overline{a})) \qquad \text{for all } \overline{a} \in A^n$$

$$\sum_{\overline{a}\in X} R^{\mathbb{A}}(\overline{a}) \leq q$$

- This is equivalent to

$$\sum_{\overline{a}\in X} R^{\mathbb{B}}(\phi(\overline{a})) \leq q,$$

a VCSP($\mathbb{B}$) instance

Homomorphism problem $\leq$ VCSP ???

- The reduction of homomorphism to VCSP works also when $\mathfrak{A}$ consists of all $(A, R^{\mathbb{A}})$ satisfying

$$\sum_{\bar{a} \in X} \alpha(\bar{a}) R^{\mathbb{A}}(\bar{a}) \leq q$$

  where $\alpha(\bar{a}) \in [0, \infty) \cap \mathbb{Q}$
- By taking copies, we can get rid of the $\alpha(\bar{a})$'s and get a VCSP($\mathbb{B}$) instance like before
- Now we just need to:
  1. Deal with negative coefficients
  2. Deal with more inequalities
  3. Deal with homomorphisms as weighted combinations of maps

Example of a power

- Take $\mathfrak{A} = \{\mathbb{A}^{(s)} \colon s \in \mathbb{Q} \cap [0, \infty)\}$ where $\mathbb{A}^{(s)} = (\{0,1\}; R^{\mathbb{A},s})$ with

$\bar{a}$	00	01	10	11
$R^{\mathbb{A},s}$	0	s	1	1

- How to define $\mathfrak{A}^2$? Pick $s_1, s_2 \in \mathbb{Q} \cap [0, \infty)$ and $\alpha \in \mathbb{Q} \cap [0,1]$. Let

$$R^{(s_1,s_2,\alpha)}((a_{11}, a_{12}), (a_{21}, a_{22})) = \alpha R^{\mathbb{A},s_1}(a_{11}, a_{21}) + (1-\alpha) R^{\mathbb{A},s_2}(a_{12}, a_{22})$$

- A few possible values:

$a_{11}a_{21}; a_{12}a_{22}$	00;00	00;01	00;10	10;11	...
$R^{(s_1,s_2,\alpha)}$	0	$(1-\alpha)s_2$	αs_1	$\alpha + (1-\alpha)s_2$	...

- A homomorphism $\mathfrak{A}^2 \to \mathfrak{A}$ can pick its favorite values s_1, s_2 (bigger is better) and α

Example, cont.

$\bar{a}$	00	01	10	11
$R^{\mathbb{A},s}$	0	s	1	1

- Effectively anything with $0?;1?$ or $?0;?1$ has infinite value
- The table of finite values

$a_{11}a_{21};a_{12}a_{22}$	00;00	10;00	10;01	01;00	01;01
$R^{(s_1,s_2,\alpha)}$	0	α	α	$1-\alpha$	$1-\alpha$

$a_{11}a_{21};a_{12}a_{22}$	11;00	11;01	11;10	11;11
$R^{(s_1,s_2,\alpha)}$	1	1	1	1

- $0.5\wedge +0.5\vee$ is $\mathfrak{A}^2 \to \mathfrak{A}$; choose $\alpha = 1/2$

$$a_1a_2 \mapsto 0.5a_1 \wedge a_2 + 0.5a_1 \vee a_2$$

- Example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto 0.5 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0.5 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

How to make our category more correct

- Sad fact: Our category has no terminal object
- Assume $\mathfrak{T}$ is a terminal object
- Say we have just one binary R in signature
- For $s \in \mathbb{Q}$ define $\mathfrak{A}^{(s)} = \{(\{0\}; R^{(s)})\}$ let $R^{(s)}(0,0) = s$
- We should have $\mathfrak{A}^{(s)} \to \mathfrak{T}$ for all $s \in \mathbb{Q}$
- Support set of T is finite; there exist only $|T|$ maps $\{0\} \to T$
- There is a map $h\colon \{0\} \to T$ that gets weight $\geq 1/|T|$ for arbitrarily small s
- Then $s \geq R^{\mathbb{T}}(h(0), h(0))$ for any $\mathbb{T} \in \mathfrak{T}$ and $s \to -\infty$

Powers $\mathfrak{B}^{\mathfrak{C}}$

- Is there right adjoint to $\mathrm{Hom}(- \times \mathfrak{C}, -)$?
- I suspect not :(