Polymorphisms of directed graphs

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Polymorphisms

- $\mathbb{A} = (A; R_1, \ldots, R_k)$
- $f \colon A^n \to A$ is a polymorphism of $\mathbb{A}$ if f is compatible with all operations of $\mathbb{A}$
- A height 1 identity for is an identity of the form

$$f(? \cdots ?) \approx g(? \cdots ?),$$

  where question marks are variables
- CSP theory: The more height 1 identities Pol($\mathbb{A}$) satisfies, the easier CSP($\mathbb{A}$) is
- L. Barto, J. Opršal, M. Pinsker, The wonderland of reflections, Israel Journal of Mathematics 223/1 (2018), 363-398
- L. Barto, J. Bulin, A. Krokhin, J. Opršal, Algebraic approach to promise constraint satisfaction

Polymorphisms of digraphs

- Directed graphs were one of the earliest objects for CSP
- $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$ has as polymorphisms all $f\colon V(\mathbb{G})^n \to V(\mathbb{G})$ such that whenever

$$\begin{matrix} u_1 & u_2 & \dots & u_n \\ \downarrow & \downarrow & \dots & \downarrow, \\ v_1 & v_2 & \dots & v_n \end{matrix}$$

then

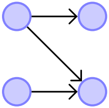
$$\begin{matrix} f(u_1, u_2, \dots, u_n) \\ \downarrow \\ f(v_1, v_2, \dots, v_n) \end{matrix}$$

Reduction to digraphs

- Feder, Vardi: Every CSP($\mathbb{A}$) is poly-time equivalent to CSP($\mathbb{G}$) for some $\mathbb{G}$ balanced digraph.
- Feder, Vardi: The Computational Structure of Monotone Monadic SNP and Constraint Satisfaction: A Study through Datalog and Group Theory, 1998
- How well do digraphs simulate polymorphisms of general relational structures?
- Bulín, Delic, Jackson, Niven: For every $\mathbb{A}$ there exists a digraph $\mathbb{G}$ such that for any* set of identities Σ we have $\mathbb{A}$ satisfies Σ iff $\mathbb{G}$ satisfies Σ
- Jakub Bulín and D. Delic and M. Jackson and T. Niven: A finer reduction of constraint problems to digraphs, 2015
- Warning: The asterisk hides a lot of technicalies...

Example identities for which the BDJN reduction works

- Necessary condition: The polymorphisms must be polymorphisms of



- Example: Majority

$$M(x,x,y) \approx M(x,y,x) \approx M(y,x,x) \approx x$$

- Non-example: Maltsev (after Anatoly Ivanovich Maltsev, 1909–1967)

$$p(x,x,y) \approx p(y,x,x) \approx y$$

Maltsev does not generally imply majority

- Maltsev & majority

$$p(x,x,y) \approx p(y,x,x) \approx y$$
$$M(x,x,y) \approx M(x,y,x) \approx M(y,x,x) \approx x$$

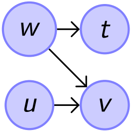
- Take $\mathbb{A} = (\{0,1\}; R)$
- $R = \{(a,b,c) \colon a+b+c = 0 \pmod 2\}$
- $p(x,y,z) = x+y+z \pmod 2$
- If M was majority, apply to rows of

$$M\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix},$$

but this is not in R

Maltsev polymorphism

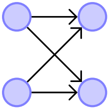
- How do Maltsev digraphs look like?
- Recall $p(x,x,y) \approx p(y,x,x) \approx y$
- If we have



- Then

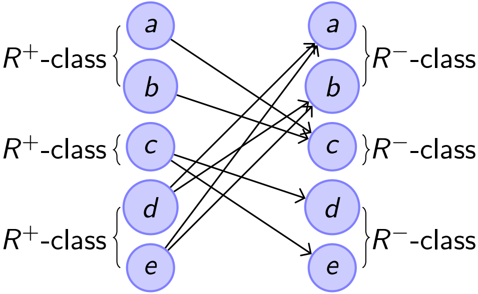
$$f\begin{pmatrix} u & w & w \\ \downarrow & \downarrow & \downarrow \\ v & v & t \end{pmatrix} = \begin{pmatrix} u \\ \downarrow \\ t \end{pmatrix},$$

- We get



Equivalence relations

- $\mathbb{G}$ be a Maltsev digraph; it looks like this:

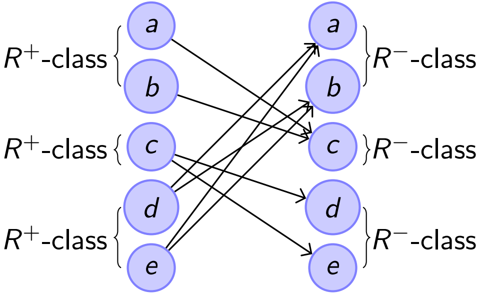


- Two partial equivalences
  1. $R^+(u, v)$ iff $\exists z$, $(u, z), (v, z) \in E(\mathbb{G})$
  2. $R^-(u, v)$ iff $\exists z$, $(z, u), (z, v) \in E(\mathbb{G})$

Factorizing a digraph

- Assume G is smooth (each vertex has in- and out-degree ≥ 1)
- R^+ and R^- are equivalences
- Equivalences factorize stuff
- Idea: Prove Maltsev $\Rightarrow$ majority by induction on digraph size, go from $\mathbb{G}/R^+$ to $\mathbb{G}$
- Induction basis: R^+, R^- are identity
- Happens for disjoint union of cycles (has Maltsev & majority)

Graphs $\mathbb{G}/R^+$ and $\mathbb{G}/R^-$



- Bijection ϕ from R^+-classes to R^--classes
- $\mathbb{G}/R^+$ and $\mathbb{G}/R^-$ turn out to be isomorphic via ϕ
- Observation: If $u \in \phi(v/R^+)$, then $v \to u$

From Maltsev to majority

- Now onto induction step
- Easy: If $\mathbb{G}$ has Maltsev, then $\mathbb{G}/R^+$ has Maltsev
- Thus $\mathbb{G}/R^+$ has majority m
- $\mathbb{G}/R^-$ has majority $m'(x,y,z) = \phi(m(\phi^{-1}(x), \phi^{-1}(y), \phi^{-1}(z)))$
- Construct a majority map $M\colon V(\mathbb{G})^3 \to V(\mathbb{G})$ so that we have

$$M(x,y,z)/R^+ = m(x/R^+, y/R^+, z/R^+)$$
$$M(x,y,z)/R^- = m'(x/R^-, y/R^-, z/R^-)$$

M is a polymorphism

- Construct a majority map $M\colon V(\mathbb{G})^3 \to V(\mathbb{G})$ so that we have

$$M(x,y,z)/R^+ = m(x/R^+, y/R^+, z/R^+)$$
$$M(x,y,z)/R^- = m'(x/R^-, y/R^-, z/R^-)$$

- Suppose we do this. Then M will be a polymorphism of $\mathbb{G}$
- Assume

$$\begin{array}{ccc} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array}$$

- Now $\phi(u_i/R^+) = v_i/R^-$ for all i
- Thus

$$M(v_1,v_2,v_3)/R^- = m'(v_1/R^-, v_2/R^-, v_3/R^-)$$
$$M(v_1,v_2,v_3)/R^- = \phi(m(\phi^{-1}(v_1/R^-), \phi^{-1}(v_2/R^-), \phi^{-1}(v_3/R^-)))$$
$$M(v_1,v_2,v_3)/R^- = \phi(m(u_1/R^+, u_2/R^+, u_3/R^+))$$
$$M(v_1,v_2,v_3)/R^- = \phi(M(u_1,u_2,u_3)/R^+)$$

Finishing the homomorphism proof

- Assume

$$\begin{array}{ccc} u_1 & u_2 & u_3 \\ \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array}$$

- We got $M(v_1, v_2, v_3)/R^- = \phi(M(u_1, u_2, u_3)/R^+)$
- By the definition of ϕ, the block $M(v_1, v_2, v_3)/R^-$ is where all the edges from $M(u_1, u_2, u_3)/R^+$ go!
- Thus $M(u_1, u_2, u_3) \to M(v_1, v_2, v_3)$ is an edge

Connections to other papers

- Maltsev $\Rightarrow$ majority shows that BDJN reduction cannot be made into a perfect correspondence
- More about Maltsev digraphs: Catarina Carvalho, Laszlo Egri, Marcel Jackson, Todd Niven. On Maltsev digraphs, 2015.
- If $\mathbb{G}$ is a Maltsev digraph, $\mathbb{G}$ has both Maltsev and majority
- This makes CSP($\mathbb{G}$) very easy, doable in logarithmic space
- This uses Victor Dalmau, Benoit Larose, Maltsev + Datalog $\Rightarrow$ symmetric Datalog, 2008