Constraint Satisfaction Problem over semilattices of Mal'cev blocks

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Motivation: Why beat a dead horse?

Dichotomy is already proved, why work on it?

We advertised the CSP as a gateway to more complex problems ("Logic for P",. . .)

Both existing proofs are complicated. Not easy to apply to a more general, more difficult problem.

Before trying that, we should simplify the Dichotomy proof.

SMB algebras

Definition

$\mathbf{A} = (A; d, \wedge)$ is a *semilattice of Mal'cev blocks* (SMB algebra) if there exists a congruence (always denoted by $\sim$) such that

- $(A/\sim; \wedge)$ is a semilattice,
- On each $\sim$-class, $\wedge$ is the second projection and
- On each $\sim$-class, $d(x, y, z)$ is a Mal'cev operation.

Each SMB algebra has a Taylor term.

SMB algebras are a quasivariety, since
$x \sim y$ iff $x \wedge y = y$ and $y \wedge x = x$.

Regular SMB algebras 1

Definition

An SMB algebra $\mathbf{A} = (A; d, \wedge)$ is *regular* if

- $x \wedge (x \wedge y) \approx y \wedge (x \wedge y) \approx (x \wedge y) \wedge y \approx x \wedge y$
- $d(x, y, z) \approx d((y \wedge z) \wedge x, (x \wedge z) \wedge y, (x \wedge y) \wedge z)$
- If $[x \wedge y]_\sim = [y]_\sim$, then $x \wedge y = y$.

Proposition

The class of all regular SMB algebras is a variety.

Proof: Syntax.

Theorem

The class of all SMB algebras is a variety.

Proof follows from: If $\mathbf{A}$ is an SMB algebra wrt $\sim$ and $\alpha \in \mathrm{Con}\, \mathbf{A}$, then $\mathbf{A}/\alpha$ is an SMB algebra wrt $(\sim \vee \alpha)/\alpha$.

Regular SMB algebras 2

Theorem

In any finite SMB algebra $\mathbf{A} = (A; d, \wedge)$ there exist terms d' and $\wedge'$ such that $(A; d', \wedge')$ is a regular SMB algebra, with respect to the same congruence $\sim$. Moreover, $(A/{\sim}; \wedge) = (A/{\sim}; \wedge')$ and $d = d'$ holds on each $\sim$-class.

Question

Can we assume that $(A; d', \wedge')$ is Taylor minimal?

It is not even known whether every finite Mal'cev algebra has a reduct which is a Taylor minimal Mal'cev algebra. We "almost" reduce to the Mal'cev case, since all term operations of a regular SMB reduct modulo $\sim$ act as the meet of their variables.

Motivation 2: SMB algebras and the colored edges theory

A. Bulatov's theory of colored edges: a TCT-like theory aimed at understanding compatible relations of a finite idempotent algebra.

Key results for Taylor algebras:

- The graph of thick edges is connected.
- There exists a Taylor reduct where all thick majority and semilattice edges are subuniverses (smooth reduct).
- The maximal elements are strongly connected in the directed thin asm-graph.
- $f(a,b) = a$ or $af(a,b)$ is a thin semilattice edge.
- ab and cd are different types of directed thin edges $\Rightarrow \exists$ a term p, $p(b,a) = b$ & $p(c,d) = d$.
- Various rectangularity properties for maximal elements.
- Quasi-2-decomposability (in SMB algebras even quasi-1-decomposability).

All trivially hold in SMB algebras.

Motivation 3: Why work on CSP for SMB algebras?

Bulatov proved first SMB algebras are tractable and then used the above results to generalize to Dichotomy.

Easier proof of tractability for SMB algebras may lead to an easier proof of the Dichotomy.

Binary CSP instances: new notation

A CSP instance is binary if all constraint relations are binary.

A binary instance P can be described with two graphs: $\Gamma_V = (V, E_V)$ and $\Gamma_P = (V_P, E_P)$ and one surjective homomorphism $pot : \Gamma_P \to \Gamma_V$.

$V = \{1, 2, \ldots, n\}$ is the set of variables, and $P_i := pot^{-1}(i)$ is the set of values which can be assigned to the variable i.

The undirected graph E_V consists of the pairs of variables between which we impose constraints.

The set of edges in E_P which maps by pot to $ij \in E_V$ is denoted by R_{ij} and $(\{i, j\}, R_{ij})$ is the constraint in the usual sense.

A solution to the instance (Γ_V, Γ_P) is a homomorphism $f : \Gamma_V \to \Gamma_P$ which satisfies $pot \circ f = id_V$.

We assume that no vertex of V is isolated in Γ_V.

We didn't do much, just converted many bipartite graphs into one multipartite graph.

Consistency and induced subinstances

Let $P = (\Gamma_V, \Gamma_P)$ be a binary CSP instance. P is 1-consistent if for all $ij \in E_V$ and all $a \in P_i$ there exists $b \in P_j$ so that $ab \in E_P$.

Let $P = (\Gamma_V, \Gamma_P)$ be a binary CSP instance. P is cycle consistent if for all cyclic graphs $C_k = 1 - 2 - \cdots - k - 1$, all $i \in V$, all homomorphisms $g : C_k \to \Gamma_V$ such that $g(1) = i$ and all $a \in P_i$ there exists a homomorphism $f : C_k \to \Gamma_P$ so that $pot \circ f = g$ and $f(1) = a$.

Let (Γ_V, Γ_P) be a binary CSP instance, $E_1 \subseteq E_V$. E_1 induces a subinstance $P' = (\Gamma_1, \Gamma_{P'})$ of P by

- $\Gamma_1 = (V_1, E_1)$, where $V_1 = \{\text{all endpoints of edges in } E_1\}$. So, Γ_1 is an edge subgraph of Γ_V with no isolated vertices.
- $\Gamma_{P'} = (V_{P'}, E_{P'})$, where $V_{P'} := pot^{-1}(V_1)$, $E_{P'} = pot^{-1}(E_1)$.

Zhuk's vocabulary

Let (Γ_V, Γ_P) be a binary CSP instance.

- (Γ_V, Γ_P) has a *subdirect solution set* if for every $a \in V_P$, $\exists$ a solution f of (Γ_V, Γ_P) such that $(f \circ pot)(a) = a$. (We can find a solution through any point of any P_i).
- (Γ_V, Γ_P) is *fragmented* if Γ_V is disconnected.
- (Γ_V, Γ_P) is *linked* if Γ_P is connected.
- (Γ_V, Γ_P) is *irreducible* if, for any $E_1 \subseteq E_V$, where $(\Gamma_1, \Gamma_{P'})$ is the subinstance of (Γ_V, Γ_P) induced by E_1, one of the following holds:
  - $(\Gamma_1, \Gamma_{P'})$ is fragmented, or
  - $(\Gamma_1, \Gamma_{P'})$ is linked, or
  - $(\Gamma_1, \Gamma_{P'})$ has a subdirect solution set.

Zhuk's Reduction Theorem

Zhuk's Reduction Theorem for SMB Algebras

Let $P = (\Gamma_V, \Gamma_P)$ be a binary CSP instance over SMB algebras. Assume that P is irreducible, 1-consistent and cycle consistent. If P has a solution, then P has a solution f such that for all $i \in V$, $f(i)$ is in the least $\sim$-class of P_i.

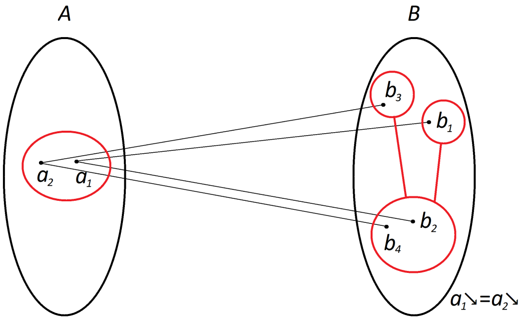
Our Reduction ~~Theorem~~ Lemma for SMB Algebras

Let $P = (\Gamma_V, \Gamma_P)$ be a binary CSP instance over SMB algebras. Assume that P is irreducible, 1-consistent, linked, but for any edge $ij \in E_V$, the subinstance induced by $E_V \setminus \{ij\}$ is not linked. If P has a solution, then P has a solution f such that for all $i \in V$, $f(i)$ is in the least $\sim$-class of P_i.

Lemmas on graphs between two SMB algebras

Lemma 1

Let $\mathbf{A}$ and $\mathbf{B}$ be two SMB algebras and $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ a subdirect product. For each $a \in A$ there is the least $\sim$-class in B which is R-connected to a, denoted by $a\searrow$. Moreover, if $a_1 \sim a_2$, then $a_1\searrow = a_2\searrow$. For $b \in B$ we use $b\swarrow$.



Lemmas on graphs between two SMB algebras 2

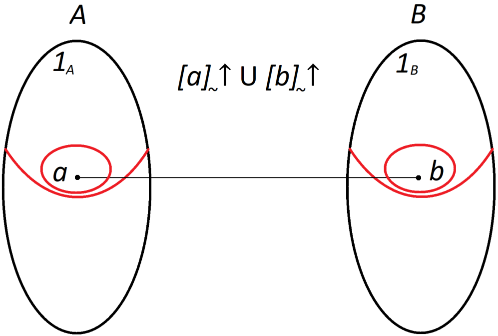
Definition

An SMB algebra $\mathbf{A}$ is *unital* if there exists some $1 \in A$ such that $1 \wedge x = x \wedge 1 = x$ holds for all $x \in A$. 1 is the unit element of $\mathbf{A}$, $[1]_\sim = \{1\}$ and it is the greatest class in the semilattice $\mathbf{A}/\sim$.

Lemmas on graphs between two SMB algebras 3

Lemma 2

Let $\mathbf{A}$ and $\mathbf{B}$ be two unital regular SMB algebras with unit elements 1_A and 1_B, and let $R \leq_{sd} \mathbf{A} \times \mathbf{B}$ be a subdirect product. The connected component of the R-graph containing 1_A also contains 1_B and it is a union of $\sim$-classes of A and of B.



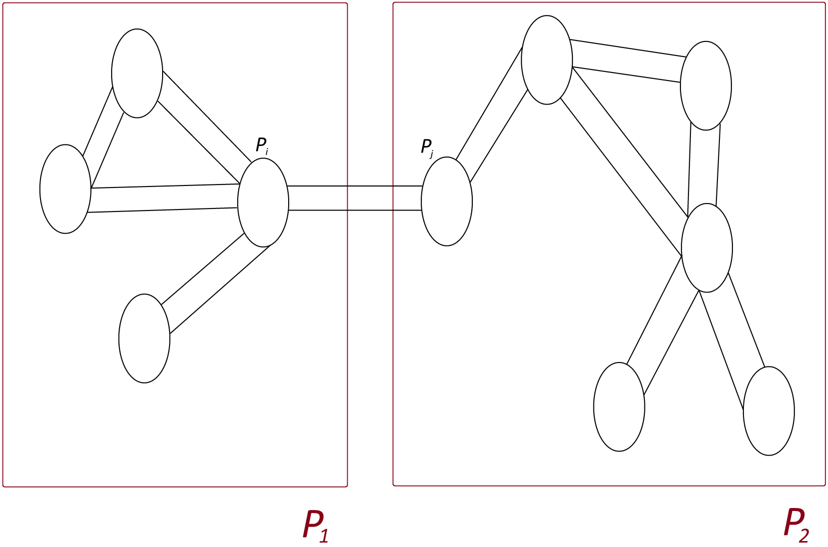
Proof of our Reduction Lemma: Bridges

We assume f is the meet of all solutions to P so f always goes through the least $\sim$-class any solution passes through.

(*) Let $ij \in E_V$ be such that $f(i)a \in R_{ij} \subseteq E_P$, where $a \in P_j$ satisfies $[a]_\sim < [f(j)]_\sim$.

Let ij be a bridge in Γ_V. The subinstance of P induced by $E_V \setminus \{ij\}$ splits into independent subinstances P_1 and P_2.

Proof of our Reduction Lemma: Bridges



Proof of our Reduction Lemma: Bridges

P_1, P_2 - not fragmented.

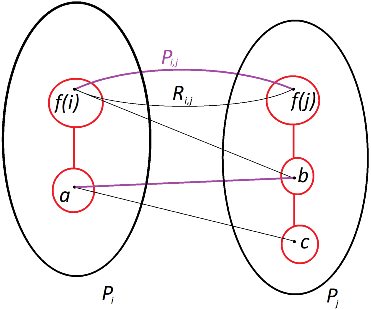
If either of P_1, P_2 is not linked, use the subdirect solution sets and, if necessary, inductive assumption (P_1 and P_2 satisfy the assumptions of our theorem if linked).

If P_1 and P_2 are both linked, then we can find another edge $i'j'$ with the same properties as ij, either in P_1 or in P_2, and reduce the search for such an edge $ij \in E_V$ to subinstance P_1 or P_2, each of which has fewer bridges than P.

Finally find $ij \in E_V$ just like in (*), but ij is not a bridge in Γ_V.

Proof of our Reduction Lemma: subdirect solution sets

Let $ij \in E_V$ be such that $f(i)a \in R_{ij} \subseteq E_P$, where $a \in P_j$ satisfies $[a]_\sim < [f(j)]_\sim$ and ij is not a bridge in Γ_V.
The subinstance P' induced by $E_V \setminus \{ij\}$ has a subdirect solution set. Two graphs on $P_i \times P_j$: $ab \in P_{ij}$ if there exists a solution g to P' such that $g(i) = a$ and $g(j) = b$. The other graph is R_{ij}. Both graphs are subdirect.



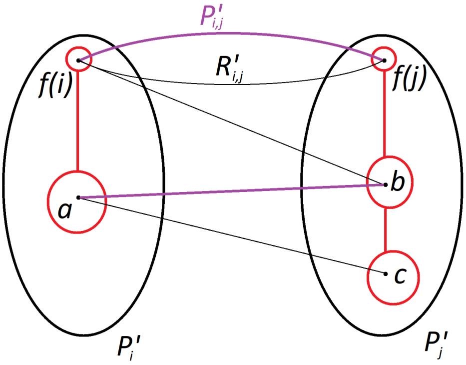
Proof of our Reduction Lemma: subdirect solution sets

Any edge in $P'_{ij} \cap R_{ij}$ - a solution to P.

Replace P_i and P_j with $P'_i = P_i \wedge f(i)$ and $P'_j = P_j \wedge f(j)$.

Restrict P'_{ij} and R_{ij} to $P'_i \times P'_j$. Both restrictions still subdirect since $f(i)f(j) \in P'_{ij} \cap R_{ij}$.

P'_i and P'_j - unital regular SMB algebras.

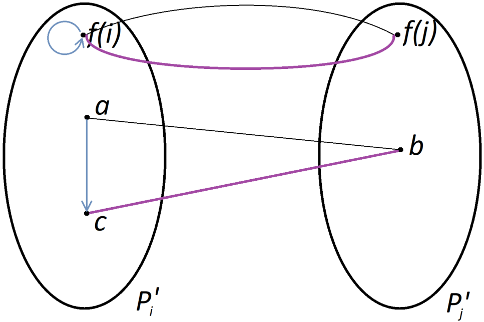


Proof of our Reduction Lemma: Loop Lemma

Use $\searrow$ in R'_{ij} and $\swarrow$ in P'_{ij}, and apply Lemma 1, to get to $a \in P'_i$, $c \in P'_j$ such that

- $f(i) \searrow\swarrow \searrow\swarrow \dots \searrow\swarrow = [a]_\sim$,
- $a\searrow = [c]_\sim$ and $c\swarrow = [a]_\sim$.

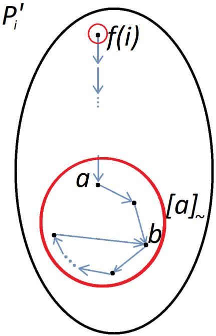
pp-define a directed graph on P'_i: $a \to c$ if $(\exists b \in P'_j) ab \in R'_{ij}$ and $bc \in P'_{ij}$.



Proof of our Reduction Lemma: Loop Lemma

May assume $f(i) \to^n a \to^m b$ and b is in the minimal (like sink) strong component A of the restriction of $\to$ to $[a]_\sim$.
Use the meet with the path from $f(i)$ to obtain that the restriction of $\to$ to A has algebraic length 1. Thus A is pp-definable $\Rightarrow$ a subuniverse.

Barto, Kozik and Niven's Loop Lemma: There is a $\to$-loop in A.
This gives a solution to P strictly below f. $\square$



Maróti reduction

Let P be a multisorted CSP instance. Under certain mild assumptions about a binary term t, M. Maróti finds a "decomposition instance" $t(P)$ with the following properties:

- If P has a solution, then so does $t(P)$.
- If $t(P)$ has a solution, this solution finds a reduction of P to an equivalent problem with at least one sort smaller.
- $t(P)$ has many more variables than P, but its sorts are all smaller (unless a "bad thing" happens).

By an inductive argument, one can solve $t(P)$ in polynomial time and thus reduce P until the "bad thing" happens in every sort.

For regular SMB algebras, "bad thing" is precisely that they are unital.

Zhuk+Maróti+consistency $\Rightarrow$ solution

Theorem 2

Let $P = (\Gamma_V, \Gamma_P)$ be a binary CSP instance over regular SMB algebras. If P is Maróti reduced, irreducible in the sense of Zhuk, 1-consistent and cycle consistent, then P has a solution.

Proof.

Maróti reduced instances are such that each P_i is unital with unit 1_i and has more than one $\sim$-class. If, for all $ij \in E_V$ one adds all edges of the form $1_i 1_j$ to R_{ij}, by Lemma 2 the connected components of induced subinstances are unchanged. Moreover, $R_{ij} \cup \{1_i 1_j\}$ are compatible. By Zhuk's Reduction Theorem, the new instance has a solution f which traverses the least $\sim$-blocks. Thus f never uses the new edges so it is also a solution to P. $\square$

Problems with Theorem 2 and improvement

Theorem 2 is useless!

Can't enforce Maróti reduced + binary instance.

Better version:

Theorem 2'

Let P be a multisorted CSP instance over regular SMB algebras. If P is Maróti reduced, irreducible in the sense of Zhuk, 1-consistent and cycle consistent, then P has a solution.

Proof.

For each constraint relation, add the tuple which is the unit element at each variable in the scope. Then apply Lemma 2 to pp-definable graphs which are used in Zhuk's original notions of connectivity of multisorted instances. Then use the same proof as Theorem 2. □

Concluding remarks

We fixed Theorem 2, but it doesn't mix well with our Reduction Lemma. One needs any arity, the other needs binary. We need to choose which way to proceed in further investigation.

THANK YOU FOR YOUR ATTENTION!