# The Amazing Power of PP Constructions

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(ongoing) joint work with Florian Starke, Albert Vucaj, Dmitriy Zhuk, ...

# Outline

- Primitive positive constructions
- **Alternative title:**
  clones on finite domains ordered by minor-preserving maps
- 2-element case, 3-element case.
- Digraphs

# Primitive Positive Constructions

Three posets on finite structures:

1. Primitive positive (pp) definability: $\underline{A} \leq_{\text{def}} \underline{B}$ if $A = B$ and every relation in $\underline{B}$ has a primitive positive definition in $\underline{A}$.

$$\exists x_1, \dots, x_n (\psi_1 \wedge \cdots \wedge \psi_m)$$

2. Primitive positive interpretations: $\underline{A} \leq_{\text{int}} \underline{B}$ if there exists $d \in \mathbb{N}$ and partial $f \colon A^d \to B$ such that preimages of relations of $\underline{B}$ are pp-definable in $\underline{A}$.
3. Primitive positive constructions (Barto, Pinsker, Opršal): $\underline{A} \leq_{\text{con}} \underline{B}$ if $\underline{B}$ is homomorphically equivalent to $\underline{B}'$ and $\underline{A} \leq_{\text{int}} \underline{B}'$.

**Motivation:**

- $\leq_{\text{def}}, \leq_{\text{int}}, \leq_{\text{con}}$ preserve the complexity of CSPs.
- Bulatov'2017, Zhuk'2017:
  CSP($\underline{B}$) is in P if $\underline{B} \not\leq_{\text{con}} K_3$, and is NP-hard otherwise.
- Relevant for: which CSPs are in L? NL? NC?
- Relevant not only for CSPs

# Posets on clones over finite sets

$\leq_{\text{def}}$, $\leq_{\text{int}}$, $\leq_{\text{con}}$: transitive.

$\text{Pol}(\underline{A})$: the clone of polymorphisms of $\underline{A}$.

1. $\underline{A} \leq_{\text{def}} \underline{B}$ iff $\text{Pol}(\underline{A}) \subseteq \text{Pol}(\underline{B})$.
2. $\underline{A} \leq_{\text{int}} \underline{B}$ iff there is a clone homomorphism $\xi \colon \text{Pol}(\underline{A}) \to \text{Pol}(\underline{B})$.

$$\xi(f(g_1, \dots, g_n)) = \xi(f)(\xi(g_1), \dots, \xi(g_n))$$
$$\xi(\pi^n_i) = \pi^n_i$$

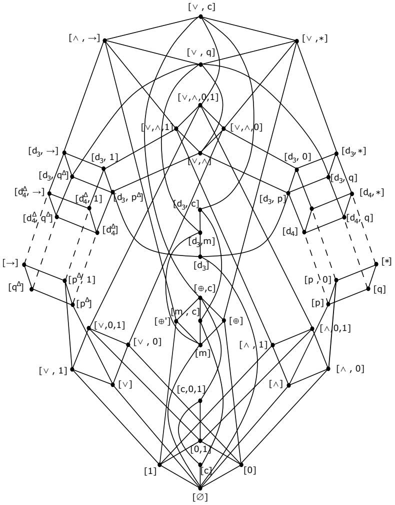
3. $\underline{A} \leq_{\text{con}} \underline{B}$ iff there is a minor-preserving map $\xi \colon \text{Pol}(\underline{A}) \to \text{Pol}(\underline{B})$.

$$\xi(f(\pi^k_{i_1}, \dots, \pi^k_{i_k})) = \xi(f)(\pi^k_{i_1}, \dots, \pi^k_{i_k})$$

(Every clone over a finite set equals $\text{Pol}(\underline{A})$ for some relational structure $\underline{A}$.)

# Clones over two elements

$\leq_{\text{def}}$ on $\{0, 1\}$:




Post's lattice

# Clones over three elements

$\leq_{\text{def}}$ on $\{0, 1, 2\}$:

Yanov-Muchnik: $2^{\omega}$

How about $\leq_{\text{int}}$?

# The interpretability poset on $\{0, 1, 2\}$

$\leq_{\text{int}}$ on $\{0, 1, 2\}$:

$$C_3 := \{(0,1),(1,2),(2,0)\}$$
$$B_2 := \{(1,0),(0,1),(1,1)\}$$
$$R_3^{=} := \{(x,y,z) \mid x \in \{0,1\} \wedge x = 0 \Rightarrow y = z\}$$

Zhuk'15: $2^\omega$ many clones between

$$\text{Pol}(\{0,1,2\}; C_3, R_3^{=})$$
$$\text{and} \quad \text{Pol}(\{0,1,2\}; C_3, B_2)$$

- Clones below $\text{Pol}(\{0,1,2\}; C_3)$: self-dual
- $\text{Pol}(\{0,1,2\}; C_3, R_3^{=})$ contains binary paper-scissor-stone operation

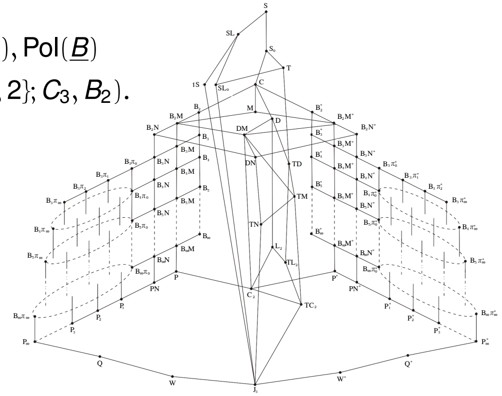
# Non-collapse

**Theorem.** Let $\underline{A}$ and $\underline{B}$ be structures s.t.

$$(\{0,1,2\}; C_3, R_3^{=}) \leq_{\text{def}} \text{Pol}(\underline{A}), \text{Pol}(\underline{B})$$
$$\leq_{\text{def}} (\{0,1,2\}; C_3, B_2).$$

If $\underline{B} \leq_{\text{int}} \underline{A}$ then $\underline{B} \leq_{\text{def}} \underline{A}$.

**Corollary:** $2^\omega$ clones over $\{0,1,2\}$ even when considered up to clone homomorphism equivalence!




**Conclusion:** Need stronger weapons.

# The constructability poset

$\leq_{\text{con}}$:

Clones $\mathcal{C}_1$ and $\mathcal{C}_2$ collapse if there is a minor-preserving map $\mathcal{C}_1 \to \mathcal{C}_2$ and a minor-preserving map $\mathcal{C}_2 \to \mathcal{C}_1$.

- all clones $\mathcal{C}$ with constant operation collapse.
- if $\mathcal{C}$ has operation with image of size $k$, then $\mathcal{C}$ collapses with a clone on $k$ elements.
- if $\mathcal{C}^{(1)} \subseteq S_n$, then $\mathcal{C}$ collapses with its idempotent reduct.
- consequence: to separate clones, can focus on idempotent strong linear Mal'cev conditions!

# The constructability poset on $\{0, 1\}$

Pieces:

all clones without cyclic operation collapse

f(x,x,y) = f(x,y,x) = f(y,x,x) = x

f(x,x,y) = f(x,y,x) = f(y,x,x) = y

f(x,y) = f(y,x)

$(\{0,1\}; B_2, \{0\}, \{1\})$ and $(\{0,1\}; \leq, \{0\}, \{1\})$ collapse (2D)

Jonsson(3)

Hagemann-Mitschke(3)

QNU(4)

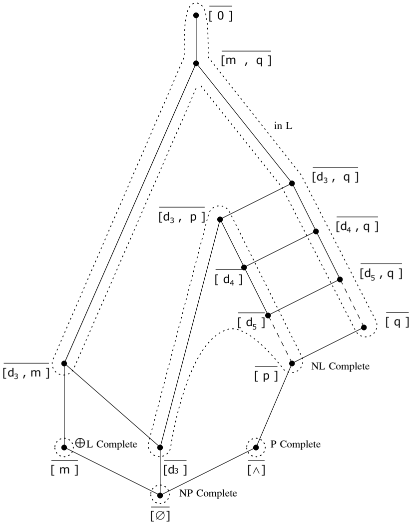
QNU(5)

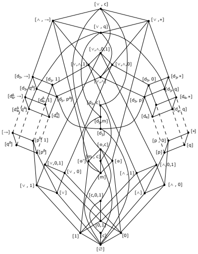
QNU(6)

...

# The constructability poset

$\leq_{\text{con}}$ on $\{0, 1\}$: outcome.



B., Vucaj 2020

# The constructability poset on $\{0, 1, 2\}$

3-4 weak near unanimity

$$f(x, x, x, y) = f(x, x, y, x) = f(x, y, x, x) = f(y, x, x, x),$$
$$f(x, x, x, y) = g(x, x, y),\ g(x, x, y) = g(x, y, x) = g(y, x, x)$$

'guarded 3-cyclic':

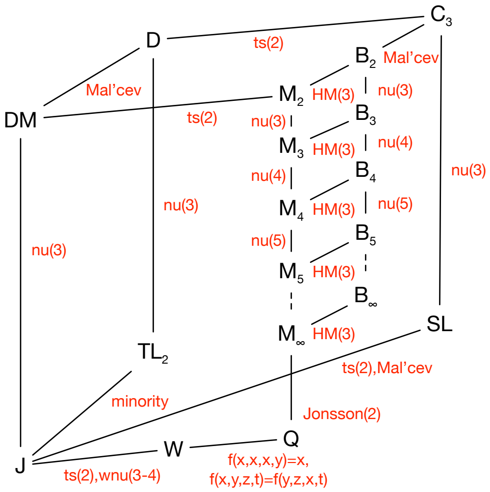
$$f(x, x, x, y) = x,\ f(x_1, x_2, x_3, y) = f(x_2, x_3, x_1, y)$$

 $(\{0, 1, 2\}; C_3, \{(x, y, z) \mid x \in \{0, 1\} \wedge x = 0 \Rightarrow y = z \in \{0, 1\}\})$ and $(\{0, 1, 2\}; C_3, \{(x, y, z) \mid x, y \in \{0, 1\} \wedge x = y = 0 \Rightarrow z = 0)$ collapse (4D).

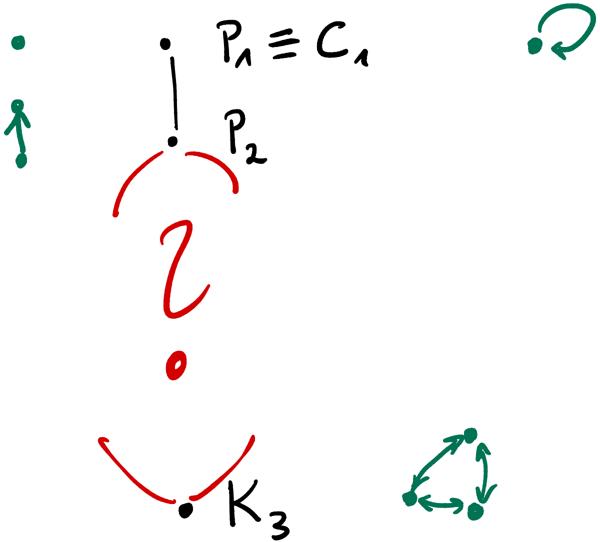
 Further collapses . . .

# The constructability poset on $\{0, 1, 2\}$

$\leq_{\text{con}}$ for self-dual clones on $\{0, 1, 2\}$: outcome.



# Digraphs



# Digraphs: pieces

$P_2 \leq_{\text{con}} D$ for every digraph $D$
with a Mal'cev polymorphism
and cyclic polymorphisms
of all prime arities

$D \leq_{\text{con}} T_3$ for every digraph $D$
without a Mal'cev polymorphism

$D \leq_{\text{con}} C_p$ for every digraph $D$
without $p$-cyclic polymorphism

$$f(y,y,x) = f(x,y,y) = y$$

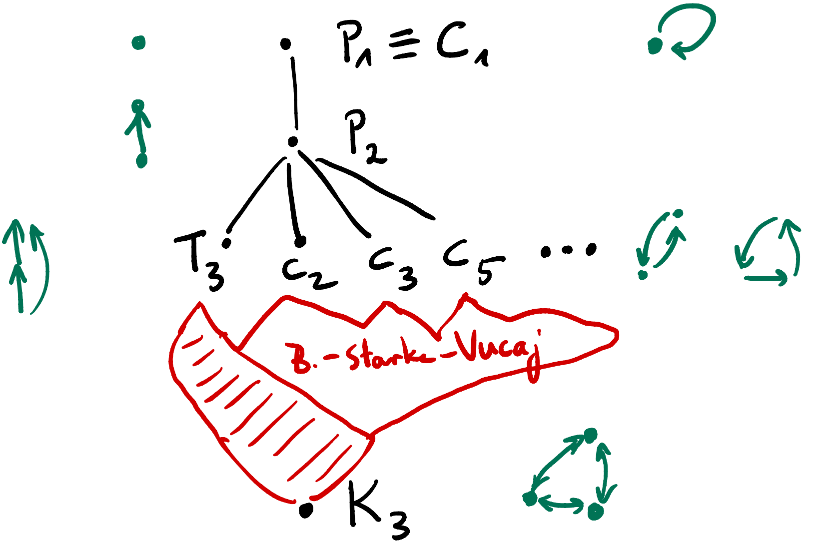
$$f(x_1,x_2) = f(x_2,x_1)$$

$$f(x_1,x_2,x_3) = f(x_2,x_3,x_1)$$

$$f(x_1,x_2,x_3,x_4) = f(x_2,x_3,x_4,x_1)$$

$$\ldots$$

# Digraphs: current state



# Recruiting

$\leq_{\text{con}}$ on finite structures:

1. What is the cardinality of $\leq_{\text{con}}$?
2. Are there infinite ascending chains?
3. Is $\leq_{\text{con}}$ a lattice?
4. What are the maximal elements below $P_2$ for general finite structures?
5. What are the maximal digraphs below $T_3$?