Stable subalgebras and weak consistency

Zarathustra Brady

Background on multisorted CSPs

- If $\mathcal{V}$ is a pseudovariety of finite algebras, we get an associated multisorted Constraint Satisfaction Problem, which we will call CSP($\mathcal{V}$).

Background on multisorted CSPs

- If $\mathcal{V}$ is a pseudovariety of finite algebras, we get an associated multisorted Constraint Satisfaction Problem, which we will call CSP($\mathcal{V}$).

- An *instance* X of CSP($\mathcal{V}$) consists of a set of *variables* and *constraints*.

Background on multisorted CSPs

- If $\mathcal{V}$ is a pseudovariety of finite algebras, we get an associated multisorted Constraint Satisfaction Problem, which we will call CSP($\mathcal{V}$).
- An *instance* X of CSP($\mathcal{V}$) consists of a set of *variables* and *constraints*.
- A *variable* is a variable name x together with a variable domain $\mathbb{A}_x \in \mathcal{V}$.

Background on multisorted CSPs

- A *constraint* is:

Background on multisorted CSPs

- A *constraint* is:
  - a *constraint relation* $\mathbb{R} \in \mathcal{V}$,

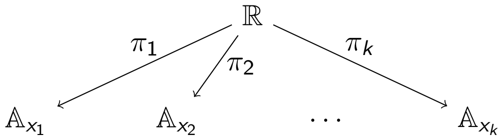
Background on multisorted CSPs

- A *constraint* is:
  - a *constraint relation* $\mathbb{R} \in \mathcal{V}$,
  - a list of variable names $x_1, \ldots, x_k$, and

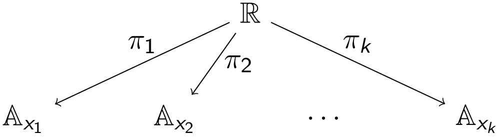
Background on multisorted CSPs

- A *constraint* is:
  - a *constraint relation* $\mathbb{R} \in \mathcal{V}$,
  - a list of variable names $x_1, \ldots, x_k$, and
  - a *projection homomorphism* $\pi_i : \mathbb{R} \to \mathbb{A}_{x_i}$ for each i.

Background on multisorted CSPs

- A *constraint* is:
  - a *constraint relation* $\mathbb{R} \in \mathcal{V}$,
  - a list of variable names $x_1, ..., x_k$, and
  - a *projection homomorphism* $\pi_i : \mathbb{R} \to \mathbb{A}_{x_i}$ for each i.
- 

Background on multisorted CSPs

- A *constraint* is:
  - a *constraint relation* $\mathbb{R} \in \mathcal{V}$,
  - a list of variable names $x_1, \ldots, x_k$, and
  - a *projection homomorphism* $\pi_i : \mathbb{R} \to \mathbb{A}_{x_i}$ for each i.
  - 

- A *solution* is an assignment $x \mapsto a_x \in \mathbb{A}_x$, such that for each constraint, $\exists r \in \mathbb{R}$ with

$$\pi_i(r) = a_{x_i}$$

for $i = 1, \ldots, k$.

Paths

- A *step* from y to z is a constraint

$$(\mathbb{R}, (x_1, ..., x_k), (\pi_1, ..., \pi_k))$$

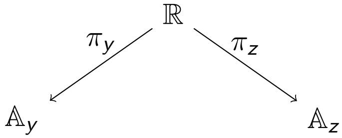
  and a pair i, j such that $x_i = y$ and $x_j = z$.

Paths

- A *step* from y to z is a constraint

$$(\mathbb{R}, (x_1, \dots, x_k), (\pi_1, \dots, \pi_k))$$

and a pair i, j such that $x_i = y$ and $x_j = z$.

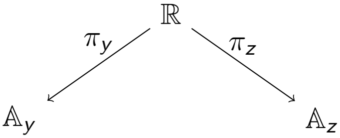
- 

Paths

- A *step* from y to z is a constraint

$$(\mathbb{R}, (x_1, ..., x_k), (\pi_1, ..., \pi_k))$$

  and a pair i, j such that $x_i = y$ and $x_j = z$.

- 

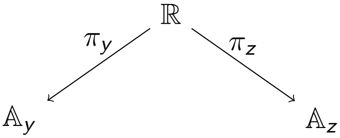
- A *path* is a sequence of steps where the endpoints match up.

Paths

- A *step* from y to z is a constraint

$$(\mathbb{R}, (x_1, ..., x_k), (\pi_1, ..., \pi_k))$$

and a pair i, j such that $x_i = y$ and $x_j = z$.

- 

- A *path* is a sequence of steps where the endpoints match up.
- We use additive notation for combining paths: $p + q$ means "first follow p, then q".

Propagating information along paths

- If $B \subseteq \mathbb{A}_y$ and p is a step from y to z through a relation $\mathbb{R}$, we write

$$B + p = B + \pi_{yz}(\mathbb{R}) = \pi_z(\pi_y^{-1}(B)) \subseteq \mathbb{A}_z.$$

Propagating information along paths

- If $B \subseteq \mathbb{A}_y$ and p is a step from y to z through a relation $\mathbb{R}$, we write

$$B + p = B + \pi_{yz}(\mathbb{R}) = \pi_z(\pi_y^{-1}(B)) \subseteq \mathbb{A}_z.$$

- This encodes the implication: "if $a_y \in B$, then $a_z \in B + p$".

Propagating information along paths

- If $B \subseteq \mathbb{A}_y$ and p is a step from y to z through a relation $\mathbb{R}$, we write

$$B + p = B + \pi_{yz}(\mathbb{R}) = \pi_z(\pi_y^{-1}(B)) \subseteq \mathbb{A}_z.$$

- This encodes the implication: "if $a_y \in B$, then $a_z \in B + p$".

- Extend this notation to paths in the obvious way:

$$B + (p_1 + p_2) = (B + p_1) + p_2, \text{ etc.}$$

Propagating information along paths

- If $B \subseteq \mathbb{A}_y$ and p is a step from y to z through a relation $\mathbb{R}$, we write

$$B + p = B + \pi_{yz}(\mathbb{R}) = \pi_z(\pi_y^{-1}(B)) \subseteq \mathbb{A}_z.$$

- This encodes the implication: "if $a_y \in B$, then $a_z \in B + p$".
- Extend this notation to paths in the obvious way:

$$B + (p_1 + p_2) = (B + p_1) + p_2, \text{ etc.}$$

- If $\mathbb{B} \leq \mathbb{A}_y$ is a subalgebra, then $\mathbb{B} + p \leq \mathbb{A}_z$ is also a subalgebra.

Consistency

- An instance is *arc-consistent* if for all paths p from x to y, we have
$$\mathbb{A}_x + p = \mathbb{A}_y.$$

Consistency

- An instance is *arc-consistent* if for all paths p from x to y, we have
$$\mathbb{A}_x + p = \mathbb{A}_y.$$
- Arc-consistency is equivalent to: for all constraint relations $\mathbb{R}$, the projections $\pi_i : \mathbb{R} \to \mathbb{A}_{x_i}$ are surjective.

Consistency

- An instance is *arc-consistent* if for all paths p from x to y, we have
$$\mathbb{A}_x + p = \mathbb{A}_y.$$
- Arc-consistency is equivalent to: for all constraint relations $\mathbb{R}$, the projections $\pi_i : \mathbb{R} \to \mathbb{A}_{x_i}$ are surjective.
- An instance is *cycle-consistent* if for all paths p from x to x, and for all $a \in \mathbb{A}_x$, we have
$$a \in \{a\} + p.$$

Consistency

- An instance is *arc-consistent* if for all paths p from x to y, we have

$$\mathbb{A}_x + p = \mathbb{A}_y.$$

- Arc-consistency is equivalent to: for all constraint relations $\mathbb{R}$, the projections $\pi_i : \mathbb{R} \to \mathbb{A}_{x_i}$ are surjective.

- An instance is *cycle-consistent* if for all paths p from x to x, and for all $a \in \mathbb{A}_x$, we have

$$a \in \{a\} + p.$$

- Beginner Sudoku players start by establishing arc-consistency, then they move on to establishing cycle-consistency.

Bounded Width

Theorem (Bulatov, Barto, Kozik)

If $\mathcal{V}$ *is a pseudo-variety of finite idempotent algebras, then TFAE:*

- $\mathrm{CSP}(\mathcal{V})$ *can be solved by a local consistency algorithm,*

Bounded Width

Theorem (Bulatov, Barto, Kozik)

If $\mathcal{V}$ is a pseudo-variety of finite idempotent algebras, then TFAE:

- $\mathrm{CSP}(\mathcal{V})$ *can be solved by a local consistency algorithm,*
- $\mathcal{V}$ *contains no nontrivial quasi-affine algebras,*

Bounded Width

Theorem (Bulatov, Barto, Kozik)

If $\mathcal{V}$ is a pseudo-variety of finite idempotent algebras, then TFAE:

- $\mathrm{CSP}(\mathcal{V})$ *can be solved by a local consistency algorithm,*
- $\mathcal{V}$ *contains no nontrivial quasi-affine algebras,*
- $\mathcal{V}$ *is congruence meet-semidistributive,*

Bounded Width

Theorem (Bulatov, Barto, Kozik)

If $\mathcal{V}$ is a pseudo-variety of finite idempotent algebras, then TFAE:

- $\mathrm{CSP}(\mathcal{V})$ *can be solved by a local consistency algorithm,*
- $\mathcal{V}$ *contains no nontrivial quasi-affine algebras,*
- $\mathcal{V}$ *is congruence meet-semidistributive,*
- *every cycle-consistent instance of* $\mathrm{CSP}(\mathcal{V})$ *has a solution.*

Some other types of consistency

- The original proof of the cycle-consistency result proved something stronger: only need "*pq*-consistency".

Some other types of consistency

- The original proof of the cycle-consistency result proved something stronger: only need "*pq*-consistency".
- An instance is *pq-consistent* if for all cycles p, q from x to x and all $a \in \mathbb{A}_x$, there exists a $j \geq 0$ such that

$$a \in \{a\} + j(p+q) + p.$$

Some other types of consistency

- The original proof of the cycle-consistency result proved something stronger: only need "*pq*-consistency".
- An instance is *pq-consistent* if for all cycles p, q from x to x and all $a \in \mathbb{A}_x$, there exists a $j \geq 0$ such that

$$a \in \{a\} + j(p + q) + p.$$

- *pq*-consistency is a strange condition, but usefully weak.

Some other types of consistency

- The original proof of the cycle-consistency result proved something stronger: only need "*pq*-consistency".
- An instance is *pq-consistent* if for all cycles p, q from x to x and all $a \in \mathbb{A}_x$, there exists a $j \geq 0$ such that

$$a \in \{a\} + j(p+q) + p.$$

- *pq*-consistency is a strange condition, but usefully weak.
- Before *pq*-consistency was introduced, there were "Prague instances".

Weak Prague Instances

- An instance is a *weak Prague instance* if:

Weak Prague Instances

- An instance is a *weak Prague instance* if:
  - (P1) it is arc-consistent,

Weak Prague Instances

- An instance is a *weak Prague instance* if:
  - (P1) it is arc-consistent,
  - (P2) $A + p = A$ implies $A - p = A$,

Weak Prague Instances

- An instance is a *weak Prague instance* if:
  - (P1) it is arc-consistent,
  - (P2) $A + p = A$ implies $A - p = A$,
  - (P3) $A + p + q = A$ implies $A + p = A$.

Weak Prague Instances

- An instance is a *weak Prague instance* if:
  - (P1) it is arc-consistent,
  - (P2) $A + p = A$ implies $A - p = A$,
  - (P3) $A + p + q = A$ implies $A + p = A$.
- Condition (P2) is closely related to the Linear Programming relaxation of the instance.

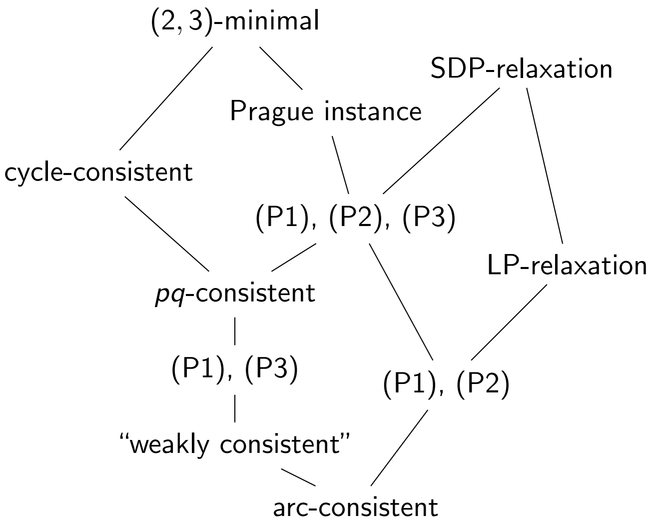
Weak Prague Instances

- An instance is a *weak Prague instance* if:
  - (P1) it is arc-consistent,
  - (P2) $A + p = A$ implies $A - p = A$,
  - (P3) $A + p + q = A$ implies $A + p = A$.
- Condition (P2) is closely related to the Linear Programming relaxation of the instance.
- Condition (P3) is closely related to the Semidefinite Programming relaxation of the instance.

Weak Prague Instances

- An instance is a *weak Prague instance* if:
  - (P1) it is arc-consistent,
  - (P2) $A + p = A$ implies $A - p = A$,
  - (P3) $A + p + q = A$ implies $A + p = A$.
- Condition (P2) is closely related to the Linear Programming relaxation of the instance.
- Condition (P3) is closely related to the Semidefinite Programming relaxation of the instance.
- Barto asks: are (P1) and (P3) enough to guarantee solvability for bounded width CSPs?

Relationships between consistency notions



Even weaker consistency!

- I call an instance *weakly consistent* if it satisfies:

  (P1) arc-consistency, and

  (W) $A + p + q = A$ implies $A \cap (A + p) \neq \emptyset$.

Even weaker consistency!

- I call an instance *weakly consistent* if it satisfies:

  (P1) arc-consistency, and

  (W) $A + p + q = A$ implies $A \cap (A + p) \neq \emptyset$.

- This is equivalent to requiring that for all cycles p, q from x to x and $a \in \mathbb{A}_x$, there exist $j, k \geq 0$ such that

$$a \in \{a\} + j(p + q) + p - k(p + q).$$

Even weaker consistency!

- I call an instance *weakly consistent* if it satisfies:

  (P1) arc-consistency, and

  (W) $A + p + q = A$ implies $A \cap (A + p) \neq \emptyset$.

- This is equivalent to requiring that for all cycles p, q from x to x and $a \in \mathbb{A}_x$, there exist $j, k \geq 0$ such that

$$a \in \{a\} + j(p + q) + p - k(p + q).$$

- My main result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite SD($\wedge$) *algebras, then every weakly consistent instance of* CSP($\mathcal{V}$) *has a solution.*

Main tool

- The only algebraic tool needed to prove this result is the concept of a *stable subalgebra*, based on the ideas in Zhuk's paper "Strong subalgebras and the Constraint Satisfaction Problem".

Main tool

- The only algebraic tool needed to prove this result is the concept of a *stable subalgebra*, based on the ideas in Zhuk's paper "Strong subalgebras and the Constraint Satisfaction Problem".

- Stable subalgebras are like absorbing subalgebras, but they are aimed at constraining the structure of *subdirect* relations instead of arbitrary relations.

Main tool

- The only algebraic tool needed to prove this result is the concept of a *stable subalgebra*, based on the ideas in Zhuk's paper "Strong subalgebras and the Constraint Satisfaction Problem".
- Stable subalgebras are like absorbing subalgebras, but they are aimed at constraining the structure of *subdirect* relations instead of arbitrary relations.
- My definition of stable subalgebras is ugly, so instead I will describe the axioms that stable subalgebras satisfy.

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.
- (Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.
- (Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.
- (Propagation) If $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is surjective, then

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.
- (Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.
- (Propagation) If $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is surjective, then
  - (Pushforward) if $\mathbb{C} \prec \mathbb{A}$, then $f(\mathbb{C}) \prec \mathbb{B}$, and

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.
- (Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.
- (Propagation) If $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is surjective, then
  - (Pushforward) if $\mathbb{C} \prec \mathbb{A}$, then $f(\mathbb{C}) \prec \mathbb{B}$, and
  - (Pullback) if $\mathbb{D} \prec \mathbb{B}$, then $f^{-1}(\mathbb{D}) \prec \mathbb{A}$.

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.
- (Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.
- (Propagation) If $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is surjective, then
  - (Pushforward) if $\mathbb{C} \prec \mathbb{A}$, then $f(\mathbb{C}) \prec \mathbb{B}$, and
  - (Pullback) if $\mathbb{D} \prec \mathbb{B}$, then $f^{-1}(\mathbb{D}) \prec \mathbb{A}$.
- (Helly) If $\mathbb{B}, \mathbb{C}, \mathbb{D} \prec \mathbb{A}$ have $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, $\mathbb{C} \cap \mathbb{D} \neq \emptyset$, and $\mathbb{B} \cap \mathbb{D} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \neq \emptyset$.

Axioms for Stability

Definition

A binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* if $\prec$ satisfies the following axioms:

- (Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \leq \mathbb{A}$.
- (Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.
- (Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.
- (Propagation) If $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is surjective, then
  - (Pushforward) if $\mathbb{C} \prec \mathbb{A}$, then $f(\mathbb{C}) \prec \mathbb{B}$, and
  - (Pullback) if $\mathbb{D} \prec \mathbb{B}$, then $f^{-1}(\mathbb{D}) \prec \mathbb{A}$.
- (Helly) If $\mathbb{B}, \mathbb{C}, \mathbb{D} \prec \mathbb{A}$ have $\mathbb{B} \cap \mathbb{C} \neq \emptyset$, $\mathbb{C} \cap \mathbb{D} \neq \emptyset$, and $\mathbb{B} \cap \mathbb{D} \neq \emptyset$, then $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \neq \emptyset$.
- (Ubiquity) For all $\mathbb{A} \in \mathcal{V}$, there is some $a \in \mathbb{A}$ such that $\{a\} \prec \mathbb{A}$.

Alternate forms of the axioms

- The propagation axiom is equivalent to:

$$\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B}, \ \ \mathbb{C} \prec \mathbb{A} \quad \Longrightarrow \quad \mathbb{C} + \mathbb{R} \prec \mathbb{B}.$$

Alternate forms of the axioms

- The propagation axiom is equivalent to:

$$\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B}, \;\; \mathbb{C} \prec \mathbb{A} \quad \Longrightarrow \quad \mathbb{C} + \mathbb{R} \prec \mathbb{B}.$$

- Modulo the intersection axiom, the Helly axiom is equivalent to:

$$\mathbb{B}_1, \ldots, \mathbb{B}_n \prec \mathbb{A}, \;\; \mathbb{B}_i \cap \mathbb{B}_j \neq \emptyset \;\; \forall i, j \quad \Longrightarrow \quad \bigcap_i \mathbb{B}_i \neq \emptyset.$$

Alternate forms of the axioms

- The propagation axiom is equivalent to:

$$\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B}, \;\; \mathbb{C} \prec \mathbb{A} \quad \Longrightarrow \quad \mathbb{C} + \mathbb{R} \prec \mathbb{B}.$$

- Modulo the intersection axiom, the Helly axiom is equivalent to:

$$\mathbb{B}_1, \ldots, \mathbb{B}_n \prec \mathbb{A}, \;\; \mathbb{B}_i \cap \mathbb{B}_j \neq \emptyset \;\; \forall i, j \quad \Longrightarrow \quad \bigcap_i \mathbb{B}_i \neq \emptyset.$$

- The propagation, intersection, and Helly axioms imply that

$$\left.\begin{array}{l} \mathbb{R} \leq_{sd} \prod_i \mathbb{A}_i \\ \mathbb{B}_i \prec \mathbb{A}_i \\ \pi_{ij}(\mathbb{R}) \cap (\mathbb{B}_i \times \mathbb{B}_j) \neq \emptyset \end{array}\right\} \Longrightarrow \mathbb{R} \cap \left(\prod_i \mathbb{B}_i\right) \neq \emptyset.$$

Stability exists

- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent SD($\wedge$) *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

Stability exists

- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent $\mathrm{SD}(\wedge)$ algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.

- My proof is an ad-hoc mess. I use König's Lemma to reduce to the case where $\mathcal{V}$ is finitely generated, take a convenient reduct...

Stability exists

- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent SD($\wedge$) *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

- My proof is an ad-hoc mess. I use König's Lemma to reduce to the case where $\mathcal{V}$ is finitely generated, take a convenient reduct...

- Morally, stability is generated by three basic cases:

Stability exists

- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent SD($\wedge$) *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

- My proof is an ad-hoc mess. I use König's Lemma to reduce to the case where $\mathcal{V}$ is finitely generated, take a convenient reduct...
- Morally, stability is generated by three basic cases:
  - Zhuk's "central" absorbing subalgebras are stable subalgebras,

Stability exists

- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent SD($\wedge$) *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

- My proof is an ad-hoc mess. I use König's Lemma to reduce to the case where $\mathcal{V}$ is finitely generated, take a convenient reduct...

- Morally, stability is generated by three basic cases:
  - Zhuk's "central" absorbing subalgebras are stable subalgebras,
  - every element of a polynomially complete, absorption-free algebra is stable, and

Stability exists

- Main technical result:

Theorem (Z.)

If $\mathcal{V}$ is a pseudovariety of finite idempotent SD($\wedge$) *algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

- My proof is an ad-hoc mess. I use König's Lemma to reduce to the case where $\mathcal{V}$ is finitely generated, take a convenient reduct...
- Morally, stability is generated by three basic cases:
  - Zhuk's "central" absorbing subalgebras are stable subalgebras,
  - every element of a polynomially complete, absorption-free algebra is stable, and
  - any subalgebra which contains a strongly absorbing subalgebra is stable.

Applying stability

- Let's use $\prec$ to prove the result about weakly consistent instances of $\mathrm{CSP}(\mathcal{V})$.

Applying stability

- Let's use $\prec$ to prove the result about weakly consistent instances of $\mathrm{CSP}(\mathcal{V})$.
- First we go even weaker...

Applying stability

- Let's use $\prec$ to prove the result about weakly consistent instances of CSP($\mathcal{V}$).
- First we go even weaker...
- Say an instance is *stably consistent* if:

(P1) it is arc-consistent, and

(S) if $\mathbb{B} \prec \mathbb{A}_x$ and $\mathbb{B} + p + q = \mathbb{B}$, then $\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset$.

Applying stability

- Let's use $\prec$ to prove the result about weakly consistent instances of CSP($\mathcal{V}$).
- First we go even weaker...
- Say an instance is *stably consistent* if:

  (P1) it is arc-consistent, and

  (S) if $\mathbb{B} \prec \mathbb{A}_x$ and $\mathbb{B} + p + q = \mathbb{B}$, then $\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset$.

- We will prove that every stably consistent instance has a solution by induction.

Standard inductive strategy

- All the arguments in the literature on bounded width CSPs have the same structure:

Standard inductive strategy

- All the arguments in the literature on bounded width CSPs have the same structure:

Step 1 produce an arc-consistent reduction with nice algebraic properties,

Standard inductive strategy

- All the arguments in the literature on bounded width CSPs have the same structure:

Step 1 produce an arc-consistent reduction with nice algebraic properties,

Step 2 prove that every arc-consistent reduction with nice algebraic properties inherits a stronger form of consistency.

Standard inductive strategy

- All the arguments in the literature on bounded width CSPs have the same structure:

Step 1 produce an arc-consistent reduction with nice algebraic properties,

Step 2 prove that every arc-consistent reduction with nice algebraic properties inherits a stronger form of consistency.

- By a *reduction*, I mean replace all of the variable domains and constraint relations of the instance by subalgebras of the original ones.

Step 1

- For Step 1, everyone uses the same main idea.

Step 1

- For Step 1, everyone uses the same main idea.
- Define an *implication digraph*, where:

Step 1

- For Step 1, everyone uses the same main idea.
- Define an *implication digraph*, where:
  - vertices are pairs $(x, \mathbb{B})$ s.t. $\mathbb{B} \subsetneq \mathbb{A}_x$, and

Step 1

- For Step 1, everyone uses the same main idea.
- Define an *implication digraph*, where:
  - vertices are pairs $(x, \mathbb{B})$ s.t. $\mathbb{B} \subsetneq \mathbb{A}_x$, and
  - for every step p from x to y, we have a directed edge from $(x, \mathbb{B})$ to $(y, \mathbb{B} + p)$.

Step 1

- For Step 1, everyone uses the same main idea.
- Define an *implication digraph*, where:
  - vertices are pairs $(x, \mathbb{B})$ s.t. $\mathbb{B} \subsetneq \mathbb{A}_x$, and
  - for every step p from x to y, we have a directed edge from $(x, \mathbb{B})$ to $(y, \mathbb{B} + p)$.
- Pick a "maximal" strongly connected component $\mathcal{C}$ of some subdigraph of the implication digraph.

Step 1

- For Step 1, everyone uses the same main idea.
- Define an *implication digraph*, where:
  - vertices are pairs $(x, \mathbb{B})$ s.t. $\mathbb{B} \subsetneq \mathbb{A}_x$, and
  - for every step p from x to y, we have a directed edge from $(x, \mathbb{B})$ to $(y, \mathbb{B} + p)$.
- Pick a "maximal" strongly connected component $\mathcal{C}$ of some subdigraph of the implication digraph.
- By ubiquity and propagation, we can restrict to the subdigraph of $(x, \mathbb{B})$ such that $\mathbb{B} \prec \mathbb{A}_x$, $\mathbb{B} \neq \mathbb{A}_x$.

Step 1

- For Step 1, everyone uses the same main idea.
- Define an *implication digraph*, where:
  - vertices are pairs $(x, \mathbb{B})$ s.t. $\mathbb{B} \subsetneq \mathbb{A}_x$, and
  - for every step p from x to y, we have a directed edge from $(x, \mathbb{B})$ to $(y, \mathbb{B} + p)$.
- Pick a "maximal" strongly connected component $\mathcal{C}$ of some subdigraph of the implication digraph.
- By ubiquity and propagation, we can restrict to the subdigraph of $(x, \mathbb{B})$ such that $\mathbb{B} \prec \mathbb{A}_x$, $\mathbb{B} \neq \mathbb{A}_x$.
- We now try to restrict $\mathbb{A}_x$ to $\mathbb{B}$ for every $(x, \mathbb{B})$ in our maximal strongly connected component $\mathcal{C}$.

Step 1, continued

- Possible problem: there could be multiple $\mathbb{B}$s with $(x, \mathbb{B}) \in \mathcal{C}$.

Step 1, continued

- Possible problem: there could be multiple $\mathbb{B}$s with $(x, \mathbb{B}) \in \mathcal{C}$.
- If $(x, \mathbb{B})$ and $(x, \mathbb{C})$ are both in $\mathcal{C}$, then there are p, q s.t.

$$\mathbb{B} + p = \mathbb{C}, \quad \mathbb{C} + q = \mathbb{B}.$$

Step 1, continued

- Possible problem: there could be multiple $\mathbb{B}$s with $(x, \mathbb{B}) \in \mathcal{C}$.
- If $(x, \mathbb{B})$ and $(x, \mathbb{C})$ are both in $\mathcal{C}$, then there are p, q s.t.

$$\mathbb{B} + p = \mathbb{C}, \quad \mathbb{C} + q = \mathbb{B}.$$

- In this case, stable consistency guarantees that

$$\mathbb{B} \cap \mathbb{C} = \mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset.$$

Step 1, continued

- Possible problem: there could be multiple $\mathbb{B}$s with $(x, \mathbb{B}) \in \mathcal{C}$.
- If $(x, \mathbb{B})$ and $(x, \mathbb{C})$ are both in $\mathcal{C}$, then there are p, q s.t.

$$\mathbb{B} + p = \mathbb{C}, \quad \mathbb{C} + q = \mathbb{B}.$$

- In this case, stable consistency guarantees that

$$\mathbb{B} \cap \mathbb{C} = \mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset.$$

- By the stronger version of the Helly axiom, we then have

$$\bigcap_{(x,\mathbb{B}) \in \mathcal{C}} \mathbb{B} \neq \emptyset.$$

Step 1, continued

- Possible problem: there could be multiple $\mathbb{B}$s with $(x, \mathbb{B}) \in \mathcal{C}$.
- If $(x, \mathbb{B})$ and $(x, \mathbb{C})$ are both in $\mathcal{C}$, then there are p, q s.t.

$$\mathbb{B} + p = \mathbb{C}, \;\; \mathbb{C} + q = \mathbb{B}.$$

- In this case, stable consistency guarantees that

$$\mathbb{B} \cap \mathbb{C} = \mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset.$$

- By the stronger version of the Helly axiom, we then have

$$\bigcap_{(x,\mathbb{B}) \in \mathcal{C}} \mathbb{B} \neq \emptyset.$$

- Looks good so far, but is this strong enough to guarantee arc-consistency?

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.
- Then we can find a proof tree which witnesses this.

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.
- Then we can find a proof tree which witnesses this.
- Consider this proof tree as an instance of CSP($\mathcal{V}$) (a subinstance of the "universal cover" of our original instance).

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.
- Then we can find a proof tree which witnesses this.
- Consider this proof tree as an instance of CSP($\mathcal{V}$) (a subinstance of the "universal cover" of our original instance).
- Let $\mathbb{R}$ be the set of solutions to this tree instance, where we don't restrict $\mathbb{A}_x$ to $\mathbb{B}$.

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.
- Then we can find a proof tree which witnesses this.
- Consider this proof tree as an instance of CSP($\mathcal{V}$) (a subinstance of the "universal cover" of our original instance).
- Let $\mathbb{R}$ be the set of solutions to this tree instance, where we don't restrict $\mathbb{A}_x$ to $\mathbb{B}$.
- $\mathbb{R}$ is subdirect in the product of the variable domains by arc-consistency.

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.
- Then we can find a proof tree which witnesses this.
- Consider this proof tree as an instance of CSP($\mathcal{V}$) (a subinstance of the "universal cover" of our original instance).
- Let $\mathbb{R}$ be the set of solutions to this tree instance, where we don't restrict $\mathbb{A}_x$ to $\mathbb{B}$.
- $\mathbb{R}$ is subdirect in the product of the variable domains by arc-consistency.
- Any pair of copies of $\mathbb{A}_x$ can be simultaneously restricted to $\mathbb{B}$ by stable consistency.

Step 1, completed

- Suppose that restricting $\mathbb{A}_x$ to $\mathbb{B}$ and enforcing arc-consistency causes a contradiction.
- Then we can find a proof tree which witnesses this.
- Consider this proof tree as an instance of CSP($\mathcal{V}$) (a subinstance of the "universal cover" of our original instance).
- Let $\mathbb{R}$ be the set of solutions to this tree instance, where we don't restrict $\mathbb{A}_x$ to $\mathbb{B}$.
- $\mathbb{R}$ is subdirect in the product of the variable domains by arc-consistency.
- Any pair of copies of $\mathbb{A}_x$ can be simultaneously restricted to $\mathbb{B}$ by stable consistency.
- By the Helly axiom, we can restrict all copies of $\mathbb{A}_x$ to $\mathbb{B}$ simultaneously.

Step 2

- For Step 2, let $+$ be addition of paths in the original instance, and let $+'$ be addition of paths in the reduced instance, and let $\mathbb{A}'_x$ be the reduced variable domains.

Step 2

- For Step 2, let $+$ be addition of paths in the original instance, and let $+'$ be addition of paths in the reduced instance, and let $\mathbb{A}'_x$ be the reduced variable domains.
- We just need to show that

$$\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset \quad \Longrightarrow \quad \mathbb{B} \cap (\mathbb{B} +' p) \neq \emptyset$$

for $\mathbb{B}' \prec \mathbb{A}'_x$.

Step 2

- For Step 2, let $+$ be addition of paths in the original instance, and let $+'$ be addition of paths in the reduced instance, and let $\mathbb{A}'_x$ be the reduced variable domains.
- We just need to show that

$$\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset \implies \mathbb{B} \cap (\mathbb{B} +' p) \neq \emptyset$$

  for $\mathbb{B}' \prec \mathbb{A}'_x$.
- Unroll the path p (duplicating vertices that occur along it multiple times):

$$\begin{array}{ccccccccccccc}
 & & \mathbb{A}_u & & & & & & \mathbb{A}_v & & \\
 & & \uparrow & & & & & & \uparrow & & \\
\mathbb{A}_x & \twoheadleftarrow & \mathbb{R}_1 & \twoheadrightarrow & \mathbb{A}_y & \twoheadleftarrow & \mathbb{R}_2 & \twoheadrightarrow & \mathbb{A}_z & \twoheadleftarrow & \mathbb{R}_3 & \twoheadrightarrow & \mathbb{A}_x \\
 & & & & & & & & & & \downarrow & & \\
 & & & & & & & & & & \mathbb{A}_w & &
\end{array}$$

Step 2, continued

- We need to show that there is a solution to the reduced path instance

$$
\begin{array}{ccccccccccccc}
 & & \mathbb{A}'_u & & & & & & & & \mathbb{A}'_v & & \\
 & & \twoheaduparrow & & & & & & & & \twoheaduparrow & & \\
\mathbb{A}'_x & \twoheadleftarrow & \mathbb{R}'_1 & \twoheadrightarrow & \mathbb{A}'_y & \twoheadleftarrow & \mathbb{R}'_2 & \twoheadrightarrow & \mathbb{A}'_z & \twoheadleftarrow & \mathbb{R}'_3 & \twoheadrightarrow & \mathbb{A}'_x \\
 & & & & & & & & & & \twoheaddownarrow & & \\
 & & & & & & & & & & \mathbb{A}'_w & &
\end{array}
$$

where the two copies of x are assigned values in $\mathbb{B}$.

Step 2, continued

- We need to show that there is a solution to the reduced path instance

$$
\begin{array}{ccccccccccccc}
 & & \mathbb{A}'_u & & & & & & \mathbb{A}'_v & & \\
 & & \twoheaduparrow & & & & & & \twoheaduparrow & & \\
\mathbb{A}'_x & \twoheadleftarrow & \mathbb{R}'_1 & \twoheadrightarrow & \mathbb{A}'_y & \twoheadleftarrow & \mathbb{R}'_2 & \twoheadrightarrow & \mathbb{A}'_z & \twoheadleftarrow & \mathbb{R}'_3 & \twoheadrightarrow & \mathbb{A}'_x \\
 & & & & & & & & & & \twoheaddownarrow & & \\
 & & & & & & & & & & \mathbb{A}'_w & &
\end{array}
$$

  where the two copies of x are assigned values in $\mathbb{B}$.
- Let $\mathbb{R}$ be the solution set to the original unrolled path instance.

Step 2, continued

- We need to show that there is a solution to the reduced path instance

$$\begin{array}{ccccccccccccc} & & \mathbb{A}'_u & & & & & & \mathbb{A}'_v & & \\ & & \uparrow & & & & & & \uparrow & & \\ \mathbb{A}'_x & \twoheadleftarrow & \mathbb{R}'_1 & \twoheadrightarrow & \mathbb{A}'_y & \twoheadleftarrow & \mathbb{R}'_2 & \twoheadrightarrow & \mathbb{A}'_z & \twoheadleftarrow & \mathbb{R}'_3 & \twoheadrightarrow & \mathbb{A}'_x \\ & & & & & & & & & & \downarrow & & \\ & & & & & & & & & & \mathbb{A}'_w & & \end{array}$$

  where the two copies of x are assigned values in $\mathbb{B}$.
- Let $\mathbb{R}$ be the solution set to the original unrolled path instance.
- Let $\mathbb{R}'$ be the solution set to the reduced path instance.

Step 2, continued

- We need to show that there is a solution to the reduced path instance

$$
\begin{array}{ccccccccccccc}
 & & \mathbb{A}'_u & & & & & & \mathbb{A}'_v & & \\
 & & \twoheaduparrow & & & & & & \twoheaduparrow & & \\
\mathbb{A}'_x & \twoheadleftarrow & \mathbb{R}'_1 & \twoheadrightarrow & \mathbb{A}'_y & \twoheadleftarrow & \mathbb{R}'_2 & \twoheadrightarrow & \mathbb{A}'_z & \twoheadleftarrow & \mathbb{R}'_3 & \twoheadrightarrow & \mathbb{A}'_x \\
 & & & & & & & & & & \twoheaddownarrow & & \\
 & & & & & & & & & & \mathbb{A}'_w & &
\end{array}
$$

  where the two copies of x are assigned values in $\mathbb{B}$.
- Let $\mathbb{R}$ be the solution set to the original unrolled path instance.
- Let $\mathbb{R}'$ be the solution set to the reduced path instance.
- Let $\mathbb{S}_1$ be the set of elements of $\mathbb{R}$ where the first copy of x is assigned a value in $\mathbb{B}$, and similarly define $\mathbb{S}_2$.

Step 2, continued

- We need to show that there is a solution to the reduced path instance

$$
\begin{array}{ccccccccccccc}
 & & \mathbb{A}'_u & & & & & & \mathbb{A}'_v & & \\
 & & \twoheaduparrow & & & & & & \twoheaduparrow & & \\
\mathbb{A}'_x & \twoheadleftarrow & \mathbb{R}'_1 & \twoheadrightarrow & \mathbb{A}'_y & \twoheadleftarrow & \mathbb{R}'_2 & \twoheadrightarrow \; \mathbb{A}'_z \; \twoheadleftarrow & \mathbb{R}'_3 & \twoheadrightarrow & \mathbb{A}'_x \\
 & & & & & & & & \twoheaddownarrow & & \\
 & & & & & & & & \mathbb{A}'_w & &
\end{array}
$$

  where the two copies of x are assigned values in $\mathbb{B}$.
- Let $\mathbb{R}$ be the solution set to the original unrolled path instance.
- Let $\mathbb{R}'$ be the solution set to the reduced path instance.
- Let $\mathbb{S}_1$ be the set of elements of $\mathbb{R}$ where the first copy of x is assigned a value in $\mathbb{B}$, and similarly define $\mathbb{S}_2$.
- Apply the Helly axiom to $\mathbb{S}_1, \mathbb{S}_2, \mathbb{R}' \prec \mathbb{R}$.

Remaining Questions

- Can we use weak consistency to improve the robust algorithm for solving bounded width CSPs due to Barto and Kozik?

Remaining Questions

- Can we use weak consistency to improve the robust algorithm for solving bounded width CSPs due to Barto and Kozik?
  - We can at least use it to improve the derandomization of the robust algorithm.

Remaining Questions

- Can we use weak consistency to improve the robust algorithm for solving bounded width CSPs due to Barto and Kozik?
  - We can at least use it to improve the derandomization of the robust algorithm.
- Is there a “canonical” stability concept?

Remaining Questions

- Can we use weak consistency to improve the robust algorithm for solving bounded width CSPs due to Barto and Kozik?
  - We can at least use it to improve the derandomization of the robust algorithm.
- Is there a "canonical" stability concept?
- How much do we have to weaken the ubiquity axiom for stability concepts in pseudovarieties which are not SD($\wedge$)?

Thank you for your attention.