Algebraic approach to the Quantified Constraint Satisfaction Problem

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Main Question

What is the complexity of QCSP(Γ) for different Γ?

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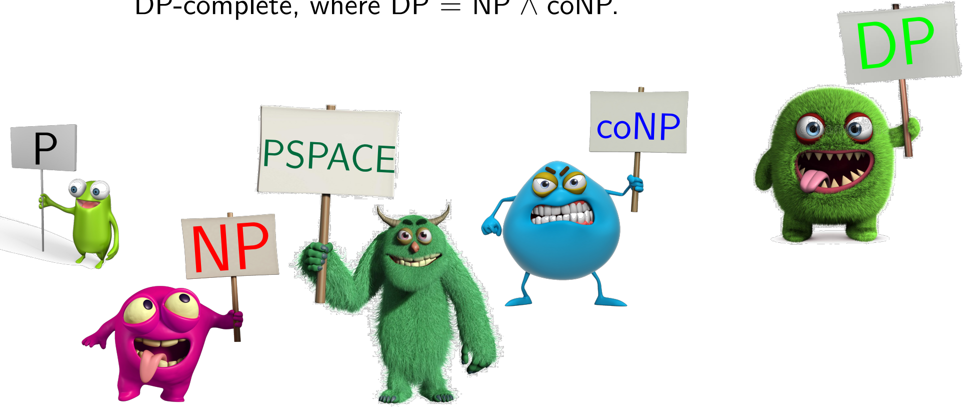

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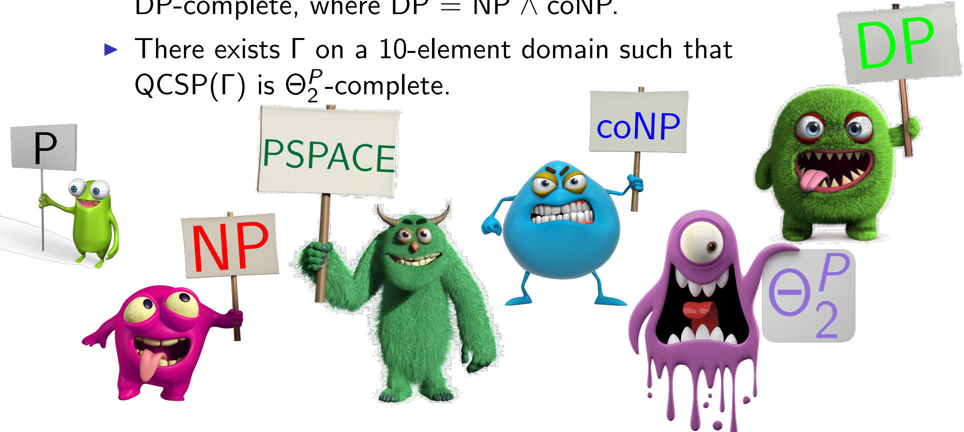
- There exists Γ on a 10-element domain such that QCSP(Γ) is Θ_2^P-complete.



Algebraic approach

- Surjective polymorphisms determine the complexity of QCSP
- How to go from PSpace to NP?
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Observation

Suppose each relation of Γ_1 is definable from Γ_2 using quantified conjunctive formulas

$$R(x_1, \ldots, x_n) = \forall y_1 \exists y_2 \forall y_3 \exists y_4 \ldots R_1(\ldots) \wedge \cdots \wedge R_s(\ldots).$$

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Theorem (Galois Correspondence, Börner, Bulatov, Chen, Jeavons, and Krokhin)

Γ_1 *is definable by quantified conjunctive formulas over* Γ_2 *IFF each surjective polymorphism of* Γ_2 *is a polymorphism of* Γ_1*, i.e.* $\mathrm{sPol}(\Gamma_1) \supseteq \mathrm{sPol}(\Gamma_2)$.

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Corollary

Suppose $\mathrm{sPol}(\Gamma_1) \supseteq \mathrm{sPol}(\Gamma_2)$. Then $QCSP(\Gamma_1)$ is polynomially reducible to $QCSP(\Gamma_2)$.

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- We need to check that for all evaluations of $x_1, \dots, x_t$ there exists a solution of the CSP $(R_1(\dots) \wedge \dots \wedge R_s(\dots))$.
- How many tuples it is sufficient to check?

PGP vs EGP

For an algebra $(A; F)$ (a set of operations F on a set A)
$d_F(n)$ is the minimal size of a generating set of A^n.

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Examples

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Pair (a_i, a_{i+1}) with $a_i \neq a_{i+1}$ is a switch in a tuple $(a_1, \dots, a_n)$.
$(0, 0, 0, 1, 2, 2, 0, 0, 0, 0)$ has 3 switches,
$(3, 3, 3, 4, 3, 3, 3, 3, 3, 3)$ has 2 switches.

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Theorem[Zhuk, 2015]

A finite algebra $\mathbf{A}$ has PGP IFF there exists k such that each $\mathbf{A}^n$ is generated by all tuples with at most k switches.

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If Pol(Γ) has PGP, then Π_2-CSP(Γ) can be polynomially reduced to CSP($\Gamma \cup \{\{a\} \mid a \in A\}$).

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Proof: the instance is equivalent to the CSP instance

$$\bigwedge_{\substack{(a_1,\ldots,a_t) \text{ with} \\ \text{at most } k \text{ switches}}} (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots) \wedge (x_1 = a_1) \wedge \cdots \wedge (x_t = a_t))$$

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Consider the equivalent instance $\mathcal{I}$ of CSP($\Gamma \cup \{\{a\} \mid a \in A\}$).

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Π_2-CSP(Γ):

Given a sentence $\forall x_1 \ldots \forall x_t \exists y_1 \ldots \exists y_q (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$, where $R_1, \ldots, R_s \in \Gamma$.
Decide whether it holds.

Theorem

If Pol(Γ) has PGP, then Π_2-CSP(Γ) is equivalent to Π_2-CSP(Γ) with $|A|$ universally quantified variables, i.e.

$$\forall z_1 \ldots \forall z_{|A|} \exists y_1 \ldots \exists y_q (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$$

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Corollary

Suppose Pol(Γ) has PGP, then Π_2-CSP(Γ) is in NP

From PSpace to NP

From PSpace to NP

$$\exists y \forall x \ \Phi$$

From PSpace to NP

$$\exists y \forall x \; \Phi$$
$$\Updownarrow$$
$$\forall x^1 \forall x^2 \ldots \forall x^{|A|} \exists y \; \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_{|A|}$$

- Φ_i is obtained from Φ by renaming x by x^i

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$$\forall x_1^1 \dots \forall x_1^{n_1} \ \forall x_2^1 \dots \forall x_2^{n_2} \dots \forall x_t^1 \dots \forall x_t^{n_t}$$
$$\exists y_1 \exists y_2^1 \dots \exists y_2^{m_2} \dots \ \exists y_t^1 \dots \exists y_t^{m_t} \ \Phi_1 \wedge \Phi_2 \wedge \cdots \wedge \Phi_q$$

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$1\;1\;1 \dots 1\;1 \quad 1\;1\;2 \dots 2 \;\dots\; 0\;0\;0\;0 \dots 0$

$$\forall x_1^1 \dots \forall x_1^{n_1} \, \forall x_2^1 \dots \forall x_2^{n_2} \dots \forall x_t^1 \dots \forall x_t^{n_t}$$
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1 1 1 . . . 1 1 1 1 2 . . . 2 . . . 0 0 0 0 . . . 0

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- We assign $x_1^1 = \dots = x_1^{n_1} = 1, \dots, x_t^1 = \dots = x_t^{n_t} = 0$

From PSpace to NP

Theorem

Suppose Pol(Γ) is k-switchable, then $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t\ \Phi$ holds IFF for every $1 \leq n_1 < n_2 < \cdots < n_k \leq t$ the sentence

$$\forall z_0 \forall z_1 \ldots \forall z_k\ \exists y_1 \ldots \exists y_{n_1} \forall x_{n_1}\ \exists y_{n_1+1} \ldots \exists y_{n_2} \forall x_{n_2}\ \ldots$$
$$\ldots \exists y_{n_{k-1}+1} \ldots \exists y_{n_k} \forall x_{n_k} \exists y_{n_k+1} \ldots \exists y_t \Phi',$$

where Φ' is obtained from Φ by renaming variables $x_{n_i+1}, \ldots, x_{n_{i+1}}$ to z_i for every i.

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Corollary 1

Suppose Pol(Γ) has PGP, then QCSP(Γ) is in NP

From PSpace to NP

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Corollary 1

Suppose Pol(Γ) has PGP, then QCSP(Γ) is in NP

Corollary 2

Suppose Pol(Γ) has PGP, then QCSP(Γ) is equivalent to Π_2-CSP(Γ) with $|A|$ universally quantified variables.

Algebraic approach

- Surjective polymorphisms determine the complexity of QCSP
- How to go from PSpace to NP?
- How to go from PSpace to coNP?
- How to prove PSpace-hardness?
- How to go from NP to P?
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How to prove PSpace-hardness?

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Let $A = \{+, -,$ 0, 1 $\}$

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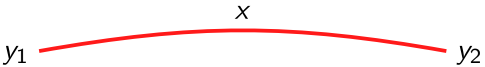

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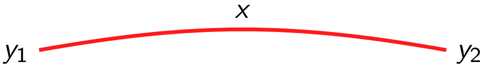

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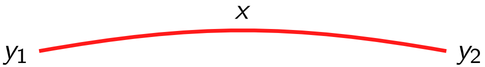

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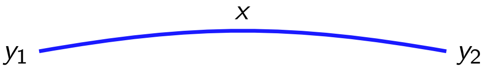

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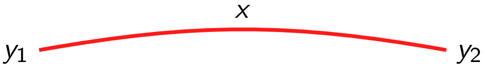

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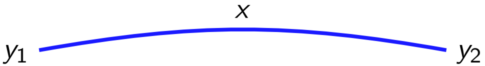
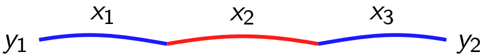


$$\exists u_1 \exists u_2 R_1(y_1, u_1, x_1) \wedge R_0(u_1, u_2, x_2) \wedge R_1(u_2, y_2, x_3)$$



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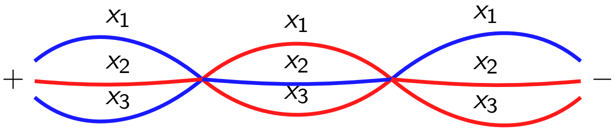

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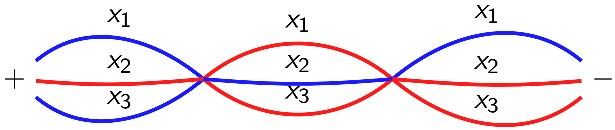

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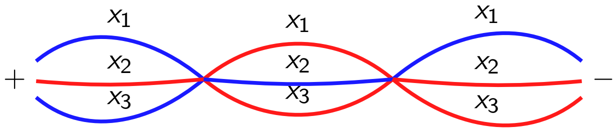

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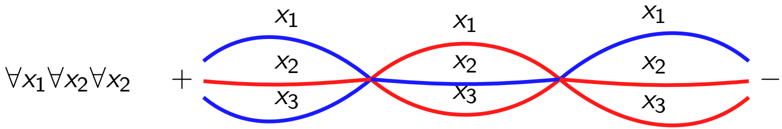

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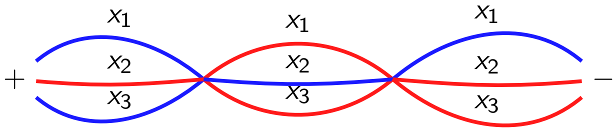

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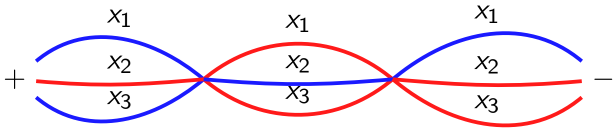

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Claim

QCSP(Γ) is coNP-hard.

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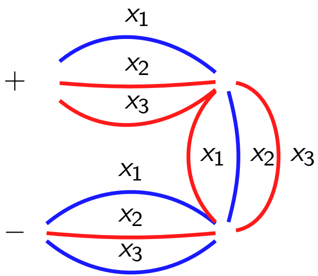

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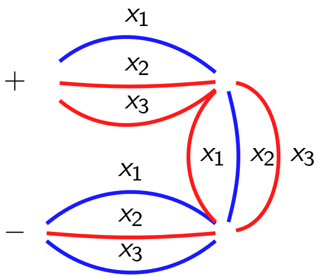

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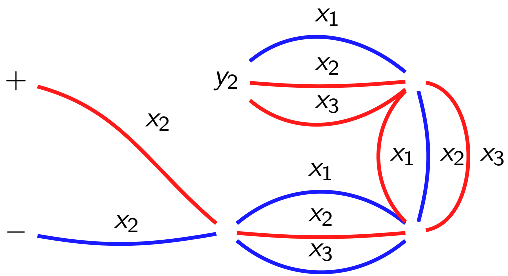

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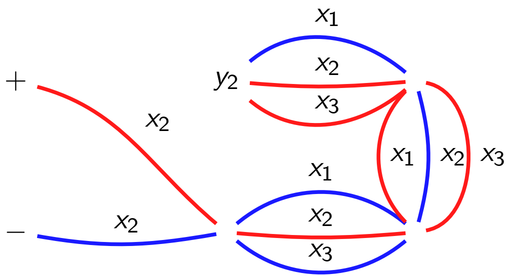

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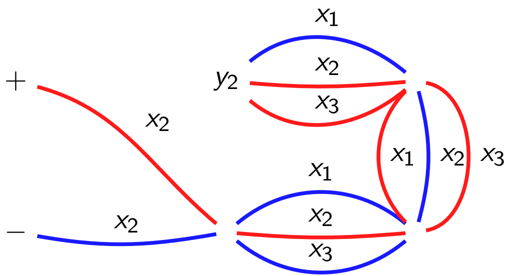
$$\neg(\exists x_1 \forall x_2 \exists x_3 \;\; ((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3))$$

How to prove PSpace-hardness?

Let $A = \{+, -, \boxed{0}, \boxed{1}\}$, $\Gamma = \{R_0, R_1, \{+\}, \{-\}\}$.

$\forall x_1 \exists y_2 \forall x_2 \forall x_2$



$$\forall x_1 \exists x_2 \forall x_3 \;\; \neg((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3))$$

$$\Updownarrow$$

$$\neg(\exists x_1 \forall x_2 \exists x_3 \;\; ((x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3))$$

Claim

QCSP(Γ) is PSpace-hard.

Algebraic approach

- Surjective polymorphisms determine the complexity of QCSP
- How to go from PSpace to NP?
- How to go from PSpace to coNP?
- How to prove PSpace-hardness?
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How to go from NP to P?

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If $\mathrm{Pol}(\Gamma)$ has PGP then

How to go from NP to P?

If Pol(Γ) has PGP then

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If $\mathrm{Pol}(\Gamma)$ has PGP then

- $\mathrm{QCSP}(\Gamma)$ can be reduced to $\mathrm{QCSP}(\Gamma)$ with bounded number of universal quantifiers
- $\mathrm{QCSP}(\Gamma)$ can be reduced to $\mathrm{CSP}(\Gamma \cup \{\{a\} \mid a \in A\})$.

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- QCSP(Γ) can be reduced to QCSP(Γ) with bounded number of universal quantifiers
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- if CSP($\Gamma \cup \{\{a\} \mid a \in A\}$) is tractable then QCSP(Γ) is tractable.

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What if $\mathrm{CSP}(\Gamma \cup \{\{a\} \mid a \in A\})$ is not tractable?

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What is the complexity for sentences $\forall x \exists y_1 \dots \exists y_t (R_1(\dots) \wedge \dots \wedge R_s(\dots))$, where $R_1, \dots, R_s \in \Gamma$.

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Γ	CSP(Γ)	∀-CSP(Γ)	CSP(Γ^*)
No-Rainbow, $\{0\}$	P	P	NPC
No-Rainbow, $\{0\}$,$\{1\}$	P	NPC	NPC

where No-Rainbow $= \{(a,b,c\} \mid |\{a,b,c\}| < 3\}$,
$\Gamma^* = \Gamma \cup \{\{a\} \mid a \in A\}$.

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- Introduce variables $\overline{y}_i = (y_i^{a_1}, \dots, y_i^{a_n})$ over the domain $A^{|A|}$.
- Replace $R_i(y_1, y_2)$ by $R_i(\overline{y}_1, \overline{y}_2) := \bigwedge_j R_i(y_1^{a_j}, y_2^{a_j})$

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- Replace $x = y_i$ by $\overline{y}_i = (a_1, \ldots, a_n)$ $\square$

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Π_2-CSP(Γ) with $|A|$ universal quantifiers is polynomially equivalent to CSP($\Gamma^{|A|^{|A|}} \cup \{U_1, \ldots, U_{|A|}\}$) for unary relations $U_1, \ldots, U_{|A|}$.

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Corollary

Suppose Pol(Γ) has PGP, then QCSP(Γ) is equivalent to CSP($\Gamma^{|A|^{|A|}} \cup \{U_1, \ldots, U_{|A|}\}$) for unary relations $U_1, \ldots, U_{|A|}$.

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Is there a better characterization than this?

Algebraic approach

- Surjective polymorphisms determine the complexity of QCSP
- How to go from PSpace to NP?
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Suppose Γ admits a WNU and contains all constants.

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a QCSP instance: $\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t \; \Phi$

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We need a polynomial algorithm to choose an assignment $y_1 = a$ s.t.

$$\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t \Phi \Longleftrightarrow \exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (\Phi \wedge y_1 = a)$$

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$$\Downarrow$$

$$\exists y_1 \forall x_1 \ldots \forall x_t \exists y_2 \ldots \exists y_t (\Phi \wedge y_1 = a)$$

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$$\Downarrow$$

$$\Phi \wedge y_1 = a \wedge x_1 = \cdots = x_t = c \text{ is satisfiable for every } c \in A$$

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$$\Downarrow$$

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$$\Downarrow$$

$$\Phi \wedge y_1 = a \wedge x_1 = \cdots = x_t = c \text{ is satisfiable for every } c \in A$$

How to choose between $y_1 = a$ and $y_2 = b$ if both satisfy this condition?

How to go from PSpace to coNP?

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$$\exists y_1 \quad \forall x_1 \quad \exists y_2 \quad \forall x_2 \quad \cdots \quad \exists y_t \quad \forall x_t \quad \Phi$$

How to go from PSpace to coNP?

$\exists y_1 \quad \forall x_1 \quad \exists y_2 \quad \forall x_2 \quad \cdots \quad \exists y_t \quad \forall x_t \quad \Phi$ $\qquad A = \{0, 1, 2\}$

How to go from PSpace to coNP?

$\exists y_1$	$\forall x_1$	$\exists y_2$	$\forall x_2$	$\cdots$	$\exists y_t$	$\forall x_t$	Φ	$A = \{0, 1, 2\}$
a	0		0	$\cdots$		0		

How to go from PSpace to coNP?

$$\exists y_1 \quad \forall x_1 \quad \exists y_2 \quad \forall x_2 \quad \cdots \quad \exists y_t \quad \forall x_t \quad \Phi \qquad\qquad A = \{0, 1, 2\}$$

a	0		0	$\cdots$		0	
a	1		1	$\cdots$		1	

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$\exists y_1 \quad \forall x_1 \quad \exists y_2 \quad \forall x_2 \quad \cdots \quad \exists y_t \quad \forall x_t \quad \Phi$ $\qquad A = \{0, 1, 2\}$

$\exists y_1$	$\forall x_1$	$\exists y_2$	$\forall x_2$	$\cdots$	$\exists y_t$	$\forall x_t$
a	0		0	$\cdots$		0
a	1		1	$\cdots$		1
a	2		2	$\cdots$		2

How to go from PSpace to coNP?

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a	0	c_2^0	0	$\cdots$	c_t^0	0	$\in \Phi$
a	1	c_2^1	1	$\cdots$	c_t^1	1	$\in \Phi$
a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$

How to go from PSpace to coNP?

$\exists y_1$	$\forall x_1$	$\exists y_2$	$\forall x_2$	$\cdots$	$\exists y_t$	$\forall x_t$	Φ
a	0	c_2^0	0	$\cdots$	c_t^0	0	$\in \Phi$
a	1	c_2^1	1	$\cdots$	c_t^1	1	$\in \Phi$
a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$
b	0		0	$\cdots$		0	

$A = \{0, 1, 2\}$

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$\exists y_1$	$\forall x_1$	$\exists y_2$	$\forall x_2$	$\cdots$	$\exists y_t$	$\forall x_t$	Φ
a	0	c_2^0	0	$\cdots$	c_t^0	0	$\in \Phi$
a	1	c_2^1	1	$\cdots$	c_t^1	1	$\in \Phi$
a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$
b	0		0	$\cdots$		0	
b	1		1	$\cdots$		1	

$A = \{0, 1, 2\}$

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a	1	c_2^1	1	$\cdots$	c_t^1	1	$\in \Phi$
a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$
b	0		0	$\cdots$		0	
b	1		1	$\cdots$		1	
b	2		2	$\cdots$		2	

How to go from PSpace to coNP?

$\exists y_1 \quad \forall x_1 \quad \exists y_2 \quad \forall x_2 \quad \cdots \quad \exists y_t \quad \forall x_t \quad \Phi$ $\qquad A = \{0, 1, 2\}$

a	0	c_2^0	0	$\cdots$	c_t^0	0	$\in \Phi$
a	1	c_2^1	1	$\cdots$	c_t^1	1	$\in \Phi$
a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$
b	0	d_2^0	0	$\cdots$	d_t^0	0	$\in \Phi$
b	1	d_2^1	1	$\cdots$	d_t^1	1	$\in \Phi$
b	2	d_2^2	2	$\cdots$	d_t^2	2	$\in \Phi$

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$\exists y_1 \quad \forall x_1 \quad \exists y_2 \quad \forall x_2 \quad \cdots \quad \exists y_t \quad \forall x_t \quad \Phi \qquad A = \{0,1,2\}$

a	0	c_2^0	0	$\cdots$	c_t^0	0	$\in \Phi$
a	1	c_2^1	1	$\cdots$	c_t^1	1	$\in \Phi$
a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$
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b	1	d_2^1	1	$\cdots$	d_t^1	1	$\in \Phi$
b	2	d_2^2	2	$\cdots$	d_t^2	2	$\in \Phi$
b	x_1	$f_2(x_1)$	x_2	$\cdots$	$f_t(x_1,\ldots,x_{t-1})$	x_t	$\in \Phi$

How to go from PSpace to coNP?

$A = \{0, 1, 2\}$

$\exists y_1$	$\forall x_1$	$\exists y_2$	$\forall x_2$	$\cdots$	$\exists y_t$	$\forall x_t$	Φ
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a	2	c_2^2	2	$\cdots$	c_t^2	2	$\in \Phi$
b	0	d_2^0	0	$\cdots$	d_t^0	0	$\in \Phi$
b	1	d_2^1	1	$\cdots$	d_t^1	1	$\in \Phi$
b	2	d_2^2	2	$\cdots$	d_t^2	2	$\in \Phi$
b	x_1	$f_2(x_1)$	x_2	$\cdots$	$f_t(x_1, \ldots, x_{t-1})$	x_t	$\in \Phi$
a	x_1	$f'_2(x_1)$	x_2	$\cdots$	$f'_t(x_1, \ldots, x_{t-1})$	x_t	$\notin \Phi$

How to go from PSpace to coNP?

$A = \{0, 1, 2\}$

$\exists y_1$	$\forall x_1$	$\exists y_2$	$\forall x_2$	$\cdots$	$\exists y_t$	$\forall x_t$	Φ
a	0		0	$\cdots$		0	$\in \Phi$
a	1		1	$\cdots$		1	$\in \Phi$
a	2		2	$\cdots$		2	$\in \Phi$
b	0		0	$\cdots$		0	$\in \Phi$
b	1		1	$\cdots$		1	$\in \Phi$
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b	x_1		x_2	$\cdots$		x_t	$\in \Phi$
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Claim 1

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Claim 2

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- Unless there is minority or majority on $\{a, b\}$ (all relations are linear or conjunction of binary relations), the problem is PSpace-hard.

Algebraic approach

- Surjective polymorphisms determine the complexity of QCSP
- How to go from PSpace to NP?
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$$\Updownarrow$$
$$\exists x_1 \ldots \exists x_{3n} (\neg (x_1 \sim x_2 \sim x_3) \wedge \cdots \wedge \neg (x_{3n-2} \sim x_{3n-1} \sim x_{3n}))$$

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- If there exists a polynomial (and efficiently computable) pp-definition of $((x_1 \sim x_2 \sim x_3) \vee \cdots \vee (x_{3n-2} \sim x_{3n-1} \sim x_{3n}))$ then QCSP(Γ) is coNP-Hard

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Conjecture

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- is tractable otherwise.

Thank you for your attention