Some Finiteness Results in Universal Algebraic Geometry via Clonoids

Patrick Wynne

University of Colorado Boulder
Ulam Seminar

April 1, 2021

Universal Algebraic Geometry

- Classical algebraic geometry studies zero-sets of multivariate polynomials.
- Universal Algebraic Geometry generalizes this notion to arbitrary algebras.
- We associate with $\mathbb{A} = (A, (f_j)_{j \in J})$ the solution set, S, of a system of algebraic equations over $\mathbb{A}$.
- $S = \{(a_1, \ldots, a_n) \in A^n \mid f_i(a_1, \ldots, a_n) = g_i(a_1, \ldots, a_n) \text{ for } i \in I\}$ where f_i and g_i are n-ary term operations of $\mathbb{A}$.
- We call such a set an **algebraic set**.

Universal Algebraic Geometry

- Two algebras defined on the same base set are said to be **algebraically equivalent** if they share the same algebraic sets.
- The intersection of algebraic sets is again algebraic.
- An algebra $\mathbb{A}$ is called an **equational domain** if for all $n \in \mathbb{N}$ and all algebraic subsets S and T of A^n, $S \cup T$ is again algebraic.
- It was shown by A.G. Pinus that on a finite set there are at most filitely many algebraically inequivalent equational domains.
- Aichinger and Rossi proved this theorem via a different method using clonoids.

Clonoids

Let A and B be sets, $k \in N$, and $f : A^k \to B$. For $n \in \mathbb{N}$ and $\sigma : [k] \to [n]$ we call the funtion f^σ a minor of f where

$$f^\sigma : A^n \to B$$

$$f^\sigma(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Clonoid

For a set A and an algebra $\mathbb{B}$ and for $C \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$ we say that C is a clonoid from A to $\mathbb{B}$ if

- C is closed under taking minors of functions, and
- for each $k \in \mathbb{N}$, $C^{[k]} := C \cap B^{A^k}$ is a subalgebra of B^{A^k}.

Definable Sets

The solution set of a term equation $s(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$ over an algebra $\mathbb{A}$ can be written in the form

$$S = \{(a_1, \ldots, a_n) \mid \mathbb{A} \models \varphi(a_1, \ldots, a_n)\}.$$

This view point more easily generalizes to arbitrary first-order structures with both functional and relational symbols.

Definition

Let $\mathbb{A} = (A, (f_i)_{i \in I}, (\rho_j)_{j \in J})$ be a first order structure, Φ a set of first-order formulas in the language of $\mathbb{A}$, and $B \subseteq A^n$. We say B is Φ-definable if there is $\varphi \in \Phi$ with free variables among $\{x_1, \ldots, x_n\}$ such that

$$B = \{(a_1, \ldots, a_n) \in A^n \mid \mathbb{A} \models \varphi(a_1, \ldots, a_n)\}.$$

We denote by $\mathrm{Def}^{[n]}(\mathbb{A}, \Phi)$ the set of Φ-definable subsets of A^n, and $\mathrm{Def}(\mathbb{A}, \Phi) = \bigcup_{n \in \mathbb{N}} \mathrm{Def}(\mathbb{A}, \Phi)$.

Minors (two more types)

We wish to study sets of formulas, Φ that are closed under taking minors in the following sense.

- We say that a formula φ is a **minor** of a formula φ' if there exists $n \in \mathbb{N}$ and $\sigma : [n] \to \mathbb{N}$ such that φ is the result of substituting free occurrences of $x_1, \ldots, x_n$ in φ' by $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$, $\varphi = \varphi'(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.
- We say $B \subseteq A^m$ is a minor of $B' \subseteq A^n$ if there exists $\sigma : [n] \to [m]$ such that

$$B = \{(a_1, \ldots, a_m) \in A^m \mid (a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \in B'\}.$$

Proposition

Let $\mathbb{A}$ be a first-order structure and let Φ be a set of first-order formulas in the language of $\mathbb{A}$ closed under $\wedge$, $\vee$, and taking minors of formulas. Then $\mathrm{Def}(\mathbb{A}, \Phi)$ is closed under finite intersections, finite unions, and taking minors of sets.

From Subsets to Functions

For a subset $T \subseteq A^n$ we define the characteristic function of T, denoted $\mathbf{1}_T$ by

$$\mathbf{1}_T : A^n \to \{0, 1\}$$

$$\mathbf{1}_T(a_1, \ldots, a_n) = \begin{cases} 1 & \text{if } (a_1, \ldots, a_n) \in T \\ 0 & \text{otherwise} . \end{cases}$$

Proposition

Let A be a finite set and let $\mathcal{R}$ be a set of finitary relations on A that is closed under finite intersections, finite unions, and taking minors of sets. Then $C(\mathcal{R}) := \{\mathbf{1}_T \mid T \in \mathcal{R}\}$ is a clonoid from A to the two-element lattice, $(\{0, 1\}, \wedge, \vee)$.

Definable Sets From Those of Bounded Arity

Theorem (A. Sparks)

Let A be a finite set and $\mathbb{B}$ a finite algebra with an n-ary near-unanimity term for some $n \geq 3$. Then for any two clonoids, C and D from A to $\mathbb{B}$, $C = D$ if and only if $C^{[|A|^{n-1}]} = D^{[|A|^{n-1}]}$.

We note that a lattice has a ternary near-unanimity term $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, and so we get the following:

Theorem

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two first-order structures on a finite set A. For each $i \in \{1, 2\}$ let Φ_i be a set of first-order formulas in the language of $\mathbb{A}_i$ that is closed under $\wedge$, $\vee$, and taking minors of formulas. Then

$$\mathrm{Def}(\mathbb{A}_1, \Phi_1) = \mathrm{Def}(\mathbb{A}_2, \Phi_2)$$

if and only if

$$\mathrm{Def}^{[|A|^2]}(\mathbb{A}_1, \Phi_1) = \mathrm{Def}^{[|A|^2]}(\mathbb{A}_2, \Phi_2)$$

Back to Universal Algebraic Geometry

- Let $\mathbb{A}$ be an algebra. The set of finite conjunctions of atomic formulas in the language of $\mathbb{A}$ is closed under $\wedge$ and taking minors of formulas, but is not closed under $\vee$ in general.
- So we cannot apply the theorem to the algebraic geometry of every finite algebra.
- We can however apply the theorem to equational domains, where the union of two algebraic sets is again algebraic.
- Fields are equational domains.

Corollary

Let A be a finite set. There are only finitely many algebraically inequivalent equational domains with universe A.

Proof:

- Let $(\mathbb{A}_i)_{i \in I}$ be a collection of algebraically inequivalent equational domains on A.
- Φ_i finite conjunctions of atomic formulas of $\mathbb{A}_i$.
- Φ_i' minimal containing Φ_i and closed under $\wedge$ and $\vee$.
- By induction on number of $\vee$ in $\varphi' \in \Phi'$, the set defined by φ' is in $\mathrm{Def}(\mathbb{A}_i, \Phi_i)$.
- Hence $\mathrm{Def}(\mathbb{A}_i, \Phi_i) = \mathrm{Def}(\mathbb{A}_i, \Phi_i')$.
- $\alpha : I \hookrightarrow \mathcal{P}(\mathcal{P}(A^{|A|^2}))$, $\alpha(i) := \mathrm{Def}^{[|A|^2]}(\mathbb{A}_i, \Phi_i')$ is injective.

Proof continued

- $\alpha : I \hookrightarrow \mathcal{P}(\mathcal{P}(A^{|A|^2}))$, $\alpha(i) := \mathrm{Def}^{[|A|^2]}(\mathbb{A}_i, \Phi'_i)$.
- Suppose $\alpha(i) = \alpha(j)$.
- $\mathrm{Def}^{[|A|^2]}(\mathbb{A}_i, \Phi'_i) = \mathrm{Def}^{[|A|^2]}(\mathbb{A}_j, \Phi'_j)$
- $\mathrm{Def}(\mathbb{A}_i, \Phi'_i) = \mathrm{Def}(\mathbb{A}_j, \Phi'_j)$ by the previous theorem.
- $\mathrm{Def}(\mathbb{A}_i, \Phi_i) = \mathrm{Def}(\mathbb{A}_j, \Phi_j)$
- So $i = j$ and α is injective.
- Hence $|I| \leq 2^{2^{|A|^{|A|^2}}}$.

Back to Universal Algebraic Geometry

Another application of this theorem comes from L_0-**logical geometry**.

- For an algebra $\mathbb{A}$, let Φ be the set of all quantifier-free formulas in the language of $\mathbb{A}$.
- Note that Φ is closed under $\vee$ as well as $\wedge$ and taking minors of formulas.
- The L_0-logical geometry of $\mathbb{A}$ is $\text{Def}(\mathbb{A}, \Phi)$.
- We say two algebras $\mathbb{A}_1$ and $\mathbb{A}_2$ on the same set A are L_0-logically equivalent if $\text{Def}(\mathbb{A}_1, \Phi_1) = \text{Def}(\mathbb{A}_2, \Phi_2)$.

Corollary

There are only finitely many L_0-logically inequivalent algebras on a finite set.

Thank You!

- This talk was based on the paper of E. Aichinger and B. Rossi titled “A clonoid based approach to some finiteness results in universal algebraic geometry.” https://link.springer.com/article/10.1007/s00012-019-0638-9
- A. Sparks, “On the number of clonoids” https://arxiv.org/abs/1810.12422

Thanks!