A theory of gadget reductions for promise constraint satisfaction

Part I

Jakub Opršal

18 Feb 2021

Durham University

overview

[KOWZ20] Andrei Krokhin, **O**, Marcin Wrochna, Standa Živný, *Topology and adjunction in promise constraint satisfaction*, arXiv:2003.11351.

[BBKO19] Libor Barto, Jakub Bulín, Andrei Krokhin, **O**, *Algebraic approach to promise constraint satisfaction*, arXiv:1811.00970v3.

an old story

- dichotomy of Boolean CSPs [Scheafer, “78]
- dichotomy of (undirected) graph CSPs [Hell, Nešetřil, “90]
- the dichotomy conjecture [Feder, Vardi, “98]
- pol-inv Galois correspondence [Cohen, Gyssens, Jeavons, “97]
- HSP closure [Bulatov, Jeavons, Krokhin, ‘05]
- Taylor implies WNU [Maróti, McKenzie, ‘08]
- algorithms given WNU polymorphisms [Bulatov, ‘17; Zhuk, ‘17]

a new story

reductions

Assume that **A** and **B** are two (finite) relational structures.

A reduction from CSP($\rho\mathbf{A}$) to CSP($\mathbf{A}$) is a mapping

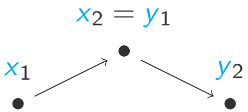
$$\lambda\colon \text{structures similar to } \rho\mathbf{A} \to \text{structures similar to } \mathbf{A}$$

such that

$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}.$$

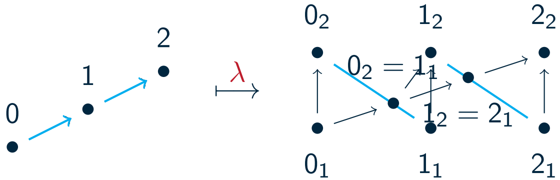
This is called adjunction.

a gadget reduction λ



$$\phi(x_1, x_2, y_1, y_2) = (x_1, x_2) \in E \wedge (y_1, y_2) \in E \wedge x_2 = y_1.$$

Example



a pp-power ρ

$\rho\mathbf{A}$ is a pp-power of $\mathbf{A}$.

Concretely, $\rho\mathbf{A} = (A^2; E^{\rho\mathbf{A}})$ where

$$
\begin{aligned}
((a_1, a_2), (b_1, b_2)) \in E^{\rho\mathbf{A}} & \\
\text{iff} \quad & \mathbf{A} \models \phi(a_1, a_2, b_1, b_2) \\
\text{iff} \quad & (a_1, a_2) \in E^{\mathbf{A}} \wedge (b_1, b_2) \in E^{\mathbf{A}} \wedge a_2 = b_1.
\end{aligned}
$$

Observation

$$\mathbf{I} \to \rho\mathbf{A} \quad \text{iff} \quad \lambda\mathbf{I} \to \mathbf{A}$$

■

gadget reductions

Let σ and τ be two relational languages. An (σ, τ)-gadget ϕ is defined by:

1. a number n,
2. a primitive positive τ-formula ϕ_R with $k \cdot n$ free variables $x_1^1, \dots, x_k^n$ for each $R \in \sigma$ of arity k.

A gadget reduction defined by such a gadget ϕ, assigns to a σ-structure $\mathbf{I}$ a structure $\lambda_\phi \mathbf{I}$ defined by:

- for each vertex $v \in I$, add to $\lambda_\phi I$ vertices $v^1, \dots, v^n$,
- for each $(v_1, \dots, v_k) \in R^{\mathbf{I}}$, ensure that

$$\lambda_\phi \mathbf{I} \models \phi^R(v_1^1, \dots, v_k^n)$$

  by adding necessary edges, or identifying vertices according to equalities in ϕ^R.

pp-powers

Let σ and τ be two relational languages. An (σ, τ)-gadget ϕ is defined by:

1. a number n,
2. a primitive positive τ-formula ϕ_R with $k \cdot n$ free variables $x_1^1, \dots, x_k^n$ for each $R \in \sigma$ of arity k.

Let $\mathbf{A}$ be a τ-structure. The pp-power of $\mathbf{A}$ defined by ϕ is the following σ-structure $\rho_\phi \mathbf{A}$.

- the universe $\rho_\phi A$ is A^n,
- $((a_1^1, \dots, a_1^n), \dots, (a_k^1, \dots, a_k^n)) \in R^{\rho \mathbf{A}}$ if

$$\mathbf{A} \models \phi^R(a_1^1, \dots, a_k^n).$$

gadget reductions and pp-powers

Observation

For all gadgets ϕ and all structures $\mathbf{I}$ and $\mathbf{A}$ of the corresponding signatures,

$$\mathbf{I} \to \rho_\phi \mathbf{A} \Leftrightarrow \lambda_\phi \mathbf{I} \to \mathbf{A}.$$

- for each $(v_1, \dots, v_k) \in R^{\mathbf{I}}$, ensure that

  $$\lambda_\phi \mathbf{I} \models \phi^R(v_1^1, \dots, v_k^n)$$

  by adding necessary edges, or identifying vertices according to equalities in ϕ^R.
- $((a_1^1, \dots, a_1^n), \dots, (a_k^1, \dots, a_k^n)) \in R^{\rho \mathbf{A}}$ if

  $$\mathbf{A} \models \phi^R(a_1^1, \dots, a_k^n).$$

algebraic approach in a nutshell

Theorem [Bulatov, Jeavons, Krokhin, '05; Barto, **O**, Pinsker, '17]
The following are equivalent for any finite relational structures $\mathbf{A}$, $\mathbf{B}$:

1. there is a gadget reduction from $\mathrm{CSP}(\mathbf{B})$ to $\mathrm{CSP}(\mathbf{A})$;
2. $\mathbf{B}$ is homomorphically equivalent to a pp-power of $\mathbf{A}$;
3. there is a minion (h1 clone) homomorphism from $\mathrm{pol}(\mathbf{A})$ to $\mathrm{pol}(\mathbf{B})$.

promises

definition of promise contraint satisfaction

Fix two finite relational structures $\mathbf{A}, \mathbf{B}$ in the same finite language with a homomorphism $\mathbf{A} \to \mathbf{B}$.

$\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ is the following problem:

Search

Given a finite structure $\mathbf{I}$ that maps homomorphically to $\mathbf{A}$, find a homomorphism $h \colon \mathbf{I} \to \mathbf{B}$.

Decide

Given $\mathbf{I}$ arbitrary structure with the same language,

- accept if $\mathbf{I} \to \mathbf{A}$,
- reject if $\mathbf{I} \not\to \mathbf{B}$.

example: 1in3- vs. NAE-Sat

- 1in3-Sat is a CSP with the template $\mathbf{T}_2 = (\{0,1\}; \text{1-in-3})$ where $\text{1-in-3} = \{(0,0,1),(0,1,0),(1,0,0)\}$.
- NAE-Sat is a CSP with the template $\mathbf{H}_2 = (\{0,1\}; \text{nae}_2)$ where $\text{nae}_2 = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$.

Clearly, $\text{1-in-3} \subseteq \text{nae}_2$, and therefore $\mathbf{T}_2 \to \mathbf{H}_2$.

The goal here is, given a solvable instance $\mathbf{I}$ of 1in3-Sat, find a solution to $\mathbf{I}$ as a NAE-Sat instance.

Both 1in3-Sat and NAE-Sat are NP-complete, but $\text{PCSP}(\mathbf{T}_2, \mathbf{H}_2)$ is in P [Brakensiek, Guruswami, '16].

reductions of promise problems

A reduction from $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$ is a mapping λ: such that

$$\mathbf{I} \to \mathbf{B}_1 \Rightarrow \lambda\mathbf{I} \to \mathbf{A}_1$$
$$\mathbf{I} \to \mathbf{B}_2 \Leftarrow \lambda\mathbf{I} \to \mathbf{A}_2.$$

Example

Assuming λ is the identity (do nothing):

$$\mathbf{I} \to \mathbf{B}_1 \Rightarrow \mathbf{I} \to \mathbf{A}_1 \quad \text{iff} \quad \mathbf{B}_1 \to \mathbf{A}_1$$
$$\mathbf{I} \to \mathbf{B}_2 \Leftarrow \mathbf{I} \to \mathbf{A}_2 \quad \text{iff} \quad \mathbf{B}_2 \leftarrow \mathbf{A}_2.$$

Definition. We say that $(\mathbf{B}_1, \mathbf{B}_2)$ is a homomorphic relaxation of $(\mathbf{A}_1, \mathbf{A}_2)$ if $\mathbf{B}_1 \to \mathbf{A}_1$ and $\mathbf{A}_2 \to \mathbf{B}_2$.

reductions of promise problems

A reduction from $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$ is a mapping λ: such that

$$\mathbf{I} \to \mathbf{B}_1 \Rightarrow \lambda\mathbf{I} \to \mathbf{A}_1$$
$$\mathbf{I} \to \mathbf{B}_2 \Leftarrow \lambda\mathbf{I} \to \mathbf{A}_2.$$

Example

Assuming λ is a gadget replacement, we have (for $i = 1, 2$)

$$\mathbf{I} \to \rho\mathbf{A}_i \Leftrightarrow \lambda\mathbf{I} \to \mathbf{A}_i$$

Therefore λ is a reduction from $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$ iff $\mathbf{B}_1 \to \rho\mathbf{A}_1$ and $\rho\mathbf{A}_2 \to \mathbf{B}_2$.

Definition. We say that $(\rho\mathbf{A}_1, \rho\mathbf{A}_2)$ is a pp-power of $(\mathbf{A}_1, \mathbf{A}_2)$.

Theorem ([Barto, Bulín, Krokhin, **O**, '19])

The following are equivalent for finite structures $\mathbf{A}_{1,2}, \mathbf{B}_{1,2}$*:*

1. *there is a gadget reduction from* $\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2)$ *to* $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$*;*
2. $(\mathbf{B}_1, \mathbf{B}_2)$ *is a homomorphic relaxation of a pp-power of* $(\mathbf{A}_1, \mathbf{A}_2)$*;*
3. *???!*

the best gadget reduction

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\lambda_1} \mathrm{PCSP}(\mathscr{P}, ?_{\mathbf{B}}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, ?_{\mathbf{A}}) \xrightarrow{\lambda_2} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

Both λ_1 and λ_2 are essentially ‘gadget reductions’. I will also describe the corresponding ‘pp-powers’.

- λ_1 and ρ_1, so that

$$\mathbf{I} \to \rho_1 \mathscr{M} \iff \lambda_1 \mathbf{I} \to \mathscr{M}$$

- λ_2 and ρ_2, so that

$$\Sigma \to \rho_2 \mathbf{A} \iff \lambda_2 \Sigma \to \mathbf{A}$$

formulation of CSP($\mathscr{P}$)

Problem

Given a minor (strong Mal'cev) condition Σ, decide whether Σ is trivial, i.e., satisfied by projections on a set of size at least 2.

A minor condition is a finite set of identities of the form

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_1, \dots, x_m)$$

for some $\pi \colon [n] \to [m]$. We often use a shorthand $f^\pi \approx g$ for the above.

ρ_2: polymorphisms

We say that $f\colon A_1^n \to A_2$ is a polymorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$ of arity n if one of the following equivalent conditions is satisfied:

- f is a homomorphism from $\mathbf{A}_1^n$ to $\mathbf{A}_2$,
- for each relation $R^{\mathbf{A}_1}$ and all tuples $\mathbf{a}_1, \dots, \mathbf{a}_n \in R^{\mathbf{A}_1}$ we have

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in R^{\mathbf{A}_2}.$$

The set of all such polymorphisms of arity n is denoted by $\mathrm{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$, and $\mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2) = \bigcup_{n\in\mathbb{N}} \mathrm{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$.

ρ_2: polymorphisms

If $f \in \mathrm{pol}^{(n)}(\mathbf{A}_1, \mathbf{A}_2)$ and $\pi\colon [n] \to [m]$, then

$$f^\pi\colon (x_1, \dots, x_n) \mapsto f(x_{\pi(1)}, \dots, x_{\pi(n)}) \in \mathrm{pol}^{(m)}(\mathbf{A}_1, \mathbf{A}_2).$$

The function f^π is called the minor of f defined by π.

A non-empty set of functions from a set A_1 to a set A_2 that is closed under taking minors is called a function minion.

- any (function) clone is a function minion.
- we say that a minor condition Σ is satisfied in $\mathscr{M}$ (and write $\Sigma \to \mathscr{M}$) if there is $\xi\colon \Sigma \to \mathscr{M}$ s.t.

$$\xi(f)^\pi = \xi(g) \text{ for each identity } f^\pi \approx g.$$

formulation of PCSP($\mathscr{M}$, $\mathscr{N}$)

Problem

Fix minion $\mathscr{M}$ and $\mathscr{N}$. Given a minor (strong Mal'cev) condition Σ,

- accept if $\Sigma \to \mathscr{M}$,
- reject if $\Sigma \not\to \mathscr{N}$.

The function minion consisting of projections on a two-element set is denoted by $\mathscr{P}$. We have $\mathscr{P} \to \mathscr{M}$ for all minions $\mathscr{M}$.

A minion homomorphism is a mapping $\xi\colon \mathscr{M} \to \mathscr{N}$ s.t.

$$\xi(f)^{\pi} = \xi(f^{\pi}) \text{ for all } \pi\colon [n] \to [m].$$

Such homomorphisms preserve satisfaction of minor conditions.

Note. Σ is trivial iff $\Sigma \to \mathscr{P}$ iff $\Sigma \to \mathscr{M}$ for all minions $\mathscr{M}$.

$\lambda_2\colon \operatorname{PCSP}(\mathscr{P}, \mathscr{M}) \to \operatorname{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$

Given a minor condition Σ, construct an instance $\mathbf{I}_{\mathbf{A}_1}(\Sigma)$ of $\operatorname{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$:

- for each symbol f of arity n in Σ, take a copy of $\mathbf{A}_1^n$ with vertices labelled by $f(a_1, \dots, a_n)$ for $a_{1,\dots,n} \in \mathbf{A}_1$;
- for each identity

$$f(x_{\pi(1)}, \dots, x_{\pi(n)}) \approx g(x_1, \dots, x_m)$$

where $\pi\colon [n] \to [m]$, and $a_{1,\dots,m} \in \mathbf{A}_1$, identify vertices labelled

$$f(a_{\pi(1)}, \dots, a_{\pi(n)}) \text{ and } g(a_1, \dots, a_m).$$

λ_2 & ρ_2: the second reduction

Observation. For all $\mathbf{C}$, we have

$$\Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{C}) \iff \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}.$$

Proof.
Assume $\xi\colon \Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{C})$ witnesses satisfcation of Σ. Define $h\colon \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}$ by

$$h\colon f(a_1, \dots, a_n) \mapsto \xi(f)(a_1, \dots, a_n).$$

Observe that (1) h is well-defined, (2) h is a homomorphism.
For the other implication, assume a homomorphism $h\colon \mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{C}$, define ξ as

$$\xi(f)\colon (a_1, \dots, a_n) = h(f(a_1, \dots, a_n)).$$

■

λ_2 & ρ_2: the second reduction

Theorem

The indicator structure gives a reduction:

$$\mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

Proof. We have that $\mathbf{I}_{\mathbf{A}_1}$ is a reduction

$$\mathrm{PCSP}(\mathrm{pol}(\mathbf{A}_1, \mathbf{A}_1), \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)) \to \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

But $\mathscr{P} \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_1)$, so we get the required reduction by homomorphic relaxation.

Alternatively, we can show directly:

1. if Σ is trivial, then $\mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{A}_1$
   —this follows since $\mathscr{P} \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_1)$, and
2. if $\mathbf{I}_{\mathbf{A}_1}(\Sigma) \to \mathbf{A}_2$ then $\Sigma \to \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)$. ∎

overview

1. If λ and ρ are adjoint, i.e., $\mathbf{A} \to \rho\mathbf{B} \Leftrightarrow \lambda\mathbf{A} \to \mathbf{B}$, then λ is a reduction from $\mathrm{PCSP}(\rho\mathbf{A}_1, \rho\mathbf{A}_2)$ to $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$.
2. we showed that $\mathbf{I}_{\mathbf{A}_1}$ and $\mathrm{pol}(\mathbf{A}_1, -)$ are adjoint.
3. this gives a reduction

$$\mathrm{PCSP}(\mathscr{P}, \mathrm{pol}(\mathbf{A}_1, \mathbf{A}_2)) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2)$$

next time...

1. Introduce λ_1, ρ_1 to complete the picture

$$\mathrm{PCSP}(\mathbf{B}_1, \mathbf{B}_2) \xrightarrow{\lambda_1} \mathrm{PCSP}(\mathscr{P}, \mathscr{B}) \xrightarrow{\mathrm{id}} \mathrm{PCSP}(\mathscr{P}, \mathscr{A}) \xrightarrow{\mathbf{I}_{\mathbf{A}_1}} \mathrm{PCSP}(\mathbf{A}_1, \mathbf{A}_2).$$

2. Show some application(s).