Sufficient conditions for a Maltsev product of two varieties to be a variety

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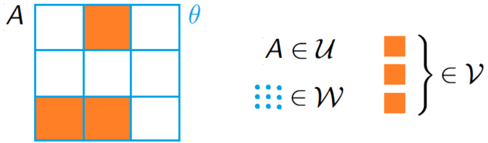
Maltsev product

For varieties $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$, the *Maltsev product* of varieties $\mathcal{V}$ and $\mathcal{W}$ *relative to* $\mathcal{U}$ is the class

$$\mathcal{V} \circ_{\mathcal{U}} \mathcal{W} = \{A \in \mathcal{U} : \exists \theta \in \mathrm{Con}(A)$$
$$A/\theta \in \mathcal{W}, \quad \forall a \in A \ \ (a/\theta \leqslant A \ \Rightarrow \ a/\theta \in \mathcal{V})\}.$$



The Maltsev product of varieties $\mathcal{V}$ and $\mathcal{W}$ of a type Ω relative to the variety of all algebras of the type Ω will be called the (*absolute*) Maltsev product of $\mathcal{V}$ and $\mathcal{W}$ and will be denoted by $\mathcal{V} \circ \mathcal{W}$.

Research problem

The class $\mathcal{V} \circ \mathcal{W}$ is closed under subalgebras, arbitrary products, and isomorphic images, but it is not in general closed under homomorphic images, and thus not a variety.

Open question: When is $\mathcal{V} \circ \mathcal{W}$ a variety?

Term idempotents

Recall that an element a of an algebra A is an *idempotent element* (or an *idempotent*) of A if for every basic operation f, one has $f(a, \ldots, a) = a$, or equivalently if $\{a\}$ is a subalgebra of A.

A term $t(x_1, \ldots, x_n)$ is a *term idempotent* of a variety $\mathcal{V}$ if for every $A \in \mathcal{V}$, the values of the corresponding term operation of A are idempotents of A.

Equivalently, $t(x_1, \ldots, x_n)$ is a term idempotent of $\mathcal{V}$ if it is an idempotent of the free algebra of $\mathcal{V}$ on generators $x_1, \ldots, x_n$.

Examples:

- The term xx^{-1} in varieties of groups or inverse semigroups.
- In an idempotent variety, every term is a term idempotent.

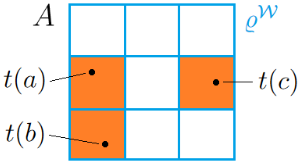
A variety $\mathcal{V}$ has a term idempotent iff every algebra in $\mathcal{V}$ has an idempotent.

The role of term idempotents

Let $\mathcal{V}$ be a variety. For every algebra A of the same type as $\mathcal{V}$, the set of congruences θ of A such that $A/\theta \in \mathcal{V}$ has the least element $\varrho^{\mathcal{V}}$. We call $A/\varrho^{\mathcal{V}}$ the *$\mathcal{V}$-replica* of A and we call $\varrho^{\mathcal{V}}$ the *$\mathcal{V}$-replica congruence* of A.

In order to find out whether an algebra A belongs to a Maltsev product $\mathcal{V} \circ \mathcal{W}$, we only need to check its $\mathcal{W}$-replica congruence: $\mathcal{V} \circ \mathcal{W} = \{A : \quad \forall a \in A \quad (a/\varrho^{\mathcal{W}} \leqslant A \;\Rightarrow\; a/\varrho^{\mathcal{W}} \in \mathcal{V})\}$.

If $t(x)$ is a term idempotent of $\mathcal{W}$, then for any algebra A, the congruence class $a/\varrho^{\mathcal{W}}$ is a subalgebra iff it contains $t(a)$ for some $a \in A$.



Equational base

If $\mathcal{W} \models p = q$, then we will say that the terms p and q are $\mathcal{W}$*-equivalent* or *equivalent in* $\mathcal{W}$.

For varieties $\mathcal{V}$ and $\mathcal{W}$, if $\Sigma_{\mathcal{V}}$ is an equational base for $\mathcal{V}$, then the variety generated by the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is defined by the following set of identities

$$\{u(p_1, \ldots, p_n) = v(p_1, \ldots, p_n) \mid (u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)) \in \Sigma_{\mathcal{V}}, \; p_1, \ldots, p_n \text{ are pairwise } \mathcal{W}\text{-equivalent term idempotents of } \mathcal{W}\},$$

Example The variety $\mathcal{L}z$ of left-zero semigroups is defined by the identity $xy = x$. The variety $\mathcal{S}$ of semilattices satisfies precisely all the identities $p = q$ such that $\mathrm{var}(p) = \mathrm{var}(q)$. Hence the variety $\mathcal{L}z \circ \mathcal{S}$ is defined by the identities $\{pq = p \mid \mathrm{var}(p) = \mathrm{var}(q)\}$.

Term idempotent varieties

A variety $\mathcal{V}$ is *term idempotent* if for every nontrivial identity $p = q$ true in $\mathcal{V}$, both terms p and q are term idempotents of $\mathcal{V}$.

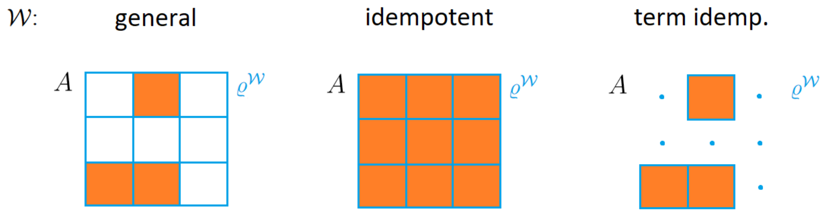
Examples:

- idempotent varieties,
- the variety of semigroups satisfying the identity $xyz = xz$,
- the variety of semigroups satisfying the identity $xyzxyz = xyz$ (this is the largest term idempotent variety contained in the variety of semigroups),
- the variety of constant semigroups (defined by the identity $xy = zt$),
- the variety of constant algebras of a given type, i.e. the algebras in which all basic operations have a common constant value.

Term idempotent varieties of a type Ω form a complete sublattice of the lattice of all varieties of the type Ω.

Characterization of term idempotent varieties

A variety $\mathcal{W}$ is term idempotent iff for every algebra A of the same type as $\mathcal{W}$, all congruence classes $a/\varrho^{\mathcal{W}}$ that are not subalgebras are singletons.



Sufficient condition

Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and let $\mathcal{W}$ be term idempotent. If there exist terms $p(x, y, z)$, $q(x, y, z)$, and $t(x)$ such that

(a) $\mathcal{V} \models p(x, y, y) = x,\ q(x, x, y) = y,$

(b) $\mathcal{W} \models p(t(x), t(x), t(y)) = q(t(x), t(y), t(y)),$

then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Remark If $\mathcal{W}$ is a term idempotent variety, then for every $A \in \mathcal{W}$, the set $\mathrm{I}(A)$ of all idempotent elements of A is a subalgebra of A. The condition (b) is equivalent to the following condition:

(b') For every $A \in \mathcal{W}$, $\mathrm{I}(A) \models p(x, x, y) = q(x, y, y)$.

Consequences

Let $\mathcal{V}$ be a congruence permutable variety and $\mathcal{W}$ be a term idempotent variety. Let $m(x, y, z)$ be a Maltsev term for $\mathcal{V}$. Define terms

$p(x, y, z) = m(x, y, z)$,

$q(x, y, z) = m(x, x, z)$,

$t(x) = x$.

Substituting these terms to conditions (a) and (b) we obtain identities that are true:

(a) $\mathcal{V} \models m(x, y, y) = x, \; m(x, x, y) = y$,

(b) $\mathcal{W} \models m(x, x, y) = m(x, x, y)$.

Consequences

Theorem

If $\mathcal{V}$ is a congruence permutable variety and $\mathcal{W}$ is a term idempotent variety, then $\mathcal{V} \circ \mathcal{W}$ is a variety.

Example The variety $\mathcal{Q}$ of quasigroups (of the type $\{\cdot, \backslash, /\}$) is congruence permutable. Let $\mathcal{W}$ be a term idempotent variety of magmas (of the type $\{\cdot\}$). We can take an equivalent variety $\mathcal{W}_3$ of the type $\{\cdot, \backslash, /\}$ with 3 binary operations that are equal (i.e. $\mathcal{W}_3 \models x \cdot y = x \backslash y = x/y$). Then $\mathcal{Q} \circ \mathcal{W}_3$ is a variety.

Consequences

A variety $\mathcal{V}$ is *congruence* 3-*permutable* if for every $A \in \mathcal{V}$ and every $\theta, \psi \in \mathrm{Con}(A)$, one has $\theta \circ \psi \circ \theta = \psi \circ \theta \circ \psi$.

Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and let $\mathcal{W}$ be term idempotent.
If $\mathcal{V} \vee \mathcal{W}$ is congruence 3-permutable, then $\mathcal{V} \circ \mathcal{W}$ is a variety.

Proof: There exist terms $p(x, y, z)$ and $q(x, y, z)$ such that

$$\mathcal{V} \vee \mathcal{W} \models x = p(x, y, y),\ p(x, x, y) = q(x, y, y),\ q(x, x, y) = y.$$

It follows that $\mathcal{V}$ and $\mathcal{W}$ also satisfy these identities, so conditions (a) and (b) are satisfied with $t(x) = x$.

Consequences

A variety $\mathcal{V}$ is *polarized* if it has a term idempotent $t(x)$ such that $\mathcal{V} \models t(x) = t(y)$. The term $t(x)$ is called a *polar term* of $\mathcal{V}$. E.g. varieties of groups are polarized with a polar term xx^{-1}.

Theorem

If $\mathcal{V}$ is a variety and $\mathcal{W}$ is a polarized term idempotent variety, then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Proof: Let $t(x)$ be a polar term of $\mathcal{W}$. Define terms

$$p(x, y, z) = x, \quad q(x, y, z) = z.$$

Substituting these terms to conditions (a) and (b) we obtain identities that are true:

(a) $\mathcal{V} \models x = x, \ y = y,$

(b) $\mathcal{W} \models t(x) = t(y).$

Ex. The variety $\mathcal{C}$ of constant algebras is polarized and term idempotent, so for any variety $\mathcal{V}$, the Maltsev product $\mathcal{V} \circ \mathcal{C}$ is a variety.

Consequences

Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and let $\mathcal{W}$ be term idempotent. If there exist terms $p(x,y)$ i $q(x,y)$ such that

(a) $\mathcal{V} \models p(x,y) = x,\ q(x,y) = y,$

(b) $\mathcal{W} \models p(x,y) = q(x,y),$

then the Maltsev product $\mathcal{V} \circ \mathcal{W}$ is a variety.

Example A group G is *Boolean* if every element of G is its own inverse, or equivalently if $G \models x^2 = e$. Let $\mathcal{B}g$ be the subvariety of the variety $\mathcal{S}g$ of semigroups defined relative to $\mathcal{S}g$ by the identities $xy^2 = x = y^2x$. Then $\mathcal{B}g$ is equivalent to the variety of Boolean groups. Let $\mathcal{R}s$ be the variety of semigroups that satisfy the identity $xyz = xz$. It is a term idempotent variety. The Maltsev product $\mathcal{B}g \circ \mathcal{R}s$ is a variety, because the conditions (a) and (b) are satisfied for terms

$$p(x,y) = xy^2, \quad q(x,y) = x^2y.$$

Consequences

Varieties $\mathcal{V}$ and $\mathcal{W}$ are *independent* if there exists a term $p(x, y)$ such that $\mathcal{V} \models p = x$ and $\mathcal{W} \models p = y$. The term p is called the *decomposition term* for $\mathcal{V}$ and $\mathcal{W}$.

Theorem

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties and let $\mathcal{W}$ be term idempotent. If $\mathcal{V}$ and $\mathcal{W}$ are independent, then $\mathcal{V} \circ \mathcal{W}$ is a variety.

Proof: Let $p(x, y)$ be a decomposition term for $\mathcal{V}$ and $\mathcal{W}$, and let $q(x, y)$ be the variable y. Substituting these terms to conditions (a) and (b) we obtain identities that are true:

(a) $\mathcal{V} \models p(x, y) = x, \ y = y,$

(b) $\mathcal{W} \models p(x, y) = y.$

Example The varieties $\mathcal{L}z$ of left-zero semigroups $(xy = x)$ and $\mathcal{R}z$ of right-zero semigroups $(xy = y)$ are independent, so $\mathcal{L}z \circ \mathcal{R}z$ is a variety.

Types of identities

An identity is

1. *regular* if it has the same variables on both sides, e.g. $xy = yx$,
2. *irregular* if the variables on its two sides differ, e.g. $xy = xx$,
3. *strongly irregular* if it is of the form $t(x, y) = x$, where the term t contains both the variables x and y, e.g. $xy = x$.

A variety is

1. *regular* if it only satisfies regular identities, e.g. the variety of semilattices,
2. *irregular* if it satisfies some irregular identity, e.g. the variety of constant semigroups,
3. *strongly irregular* if it satisfies some strongly irregular identity, e.g. the variety of groups ($xyy^{-1} = x$) or the variety of lattices ($x \vee (x \wedge y) = x$).

Ω-semilattices

For a given type Ω, let $\mathcal{S}$ denote the variety defined by all the regular identities of the type Ω. If Ω has no symbols of constants and it has at least one symbol of at least binary basic operation, then $\mathcal{S}$ is the unique variety of the type Ω which is equivalent to the variety of semilattices. The algebras in $\mathcal{S}$ are called Ω-semilattices.

For a given variety $\mathcal{V}$, algebras in the Maltsev product $\mathcal{V} \circ \mathcal{S}$ are called *semilattice sums* of algebras in $\mathcal{V}$.

Consequences

Theorem

If $\mathcal{V}$ is a strongly irregular variety, then the class $\mathcal{V} \circ \mathcal{S}$ of semilattice sums of algebras in $\mathcal{V}$ is a variety.

Proof: Let $t(x, y) = x$ be a strongly irregular identity satisfied in $\mathcal{V}$. Define terms

$$p(x, y) = t(x, y), \quad q(x, y) = t(y, x).$$

Substituting these terms to conditions (a) and (b) we obtain identities that are true:

(a) $\mathcal{V} \models t(x, y) = x, \ t(y, x) = y,$

(b) $\mathcal{S} \models t(x, y) = t(y, x).$

Example Let $\mathcal{L}$ be a variety of lattices. Then $\mathcal{L} \circ \mathcal{S}$ is a variety.

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