Saturated free algebras and almost indiscernible theories

Algebra Universalis, 2022

T. G. Kucera[1] A. Pillay[2]

[1]Department of Mathematics, University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2

[2]Department of Mathematics, University of Notre Dame
Notre Dame, Indiana 46556, USA

Panglobal Algebra and Logic Seminar, U. Colorado Boulder
2022-04-12

Saturated free algebras

1. J. T. Baldwin and S. Shelah. The structure of saturated free algebras. *Algebra Universalis*, 17(2):191–199, 1983.
2. Anand Pillay and Rizos Sklinos. Saturated free algebras revisited. *Bull. Symb. Log.*, 21(3):306–318, 2015.

Saturated free algebras

1. J. T. Baldwin and S. Shelah. The structure of saturated free algebras. *Algebra Universalis*, 17(2):191–199, 1983.
2. Anand Pillay and Rizos Sklinos. Saturated free algebras revisited. *Bull. Symb. Log.*, 21(3):306–318, 2015.

Pillay and I extend the work reported in these two papers to uncountable languages and indiscernible sets of infinite tuples.

3. Thomas G. Kucera and Anand Pillay, Saturated free algebras and almost indiscernible theories, *Algebra Universalis*, 83 (to appear), 2022.

https://doi.org/10.1007/s00012-021-00766-x

Main ideas. . .

- Universal Algebra
  - free algebras
  - varieties of algebras
- Model Theory
  - applications of core methods of stability theory
  - saturated models
  - indiscernible sets
- Applications to and examples from modules.
- Exploit *analogies* more than strict interpretations.

Main results (informal statements). . .

Definition

A complete theory T is almost indiscernible if some saturated model M, $|M| = \kappa > |T|$, is in the algebraic closure of an indiscernible set of sequences $I = \{ \underline{e}_\alpha : \alpha < \kappa \}$.

- T is superstable, stable in every $\lambda \geq |T|$, and non-multidimensional.
- For $\lambda \geq \omega$ set $M_\lambda = \text{acl}(\langle \underline{e}_\alpha : \alpha < \lambda \rangle)$.
  This is an elementary chain, with each M_λ being $F^a_{|\lambda|}$-saturated.
- There is a cardinal $\overline{\mu} \leq |T|^+$ such that every $M \succeq M_{\overline{\mu}}$ is in the algebraic closure of $M_{\overline{\mu}}$ together with realizations of an independent (over $M_{\overline{\mu}}$) weight one types

Main results (informal statements)...

Modules

A complete theory T of modules is almost indiscernible iff it is superstable with $\lambda(T) = |T|$.

Main results (informal statements)...

Modules

A complete theory T of modules is almost indiscernible iff it is superstable with $\lambda(T) = |T|$.

Free algebras

Let M be a saturated free algebra in a language of any cardinality.

- There is a stationary one-type p_0 over $\emptyset$ such that $a \in M$ is an element of a free basis for M iff a satisfies p_0.
- For $\overline{a}, \overline{b} \in M$, $\overline{a} \perp \overline{b}$ iff for some basis $A \dot{\cup} B$ of M, $\overline{a} \in \mathrm{acl}(A)$, $\overline{b} \in \mathrm{acl}(B)$.

Tools of model theory

Languages, formulas, structures . . .

- A finitary first order language $\mathcal{L}$ consisting of function symbols, predicate symbols, and constant symbols, and a supply of variables.
- Formulas of $\mathcal{L}$ created by forming terms, equations, relations, and by combining them with propositional connectives and individual quantifiers.
- A structure for $\mathcal{L}$ is a set together with operations, relations, and individual elements interpreting the symbols of $\mathcal{L}$.
- A theory T is a set of sentences of $\mathcal{L}$. T is complete if it is the set of all $\mathcal{L}$-sentences true in some $\mathcal{L}$-structure M, in which case we write $T = \mathrm{Th}(M)$.

. . . definable sets, types

- A definable set [over A] is the solution set (in some structure) of an $\mathcal{L}$-formula [with parameters from A].
- A type over A is a maximal consistent set of formulas with parameters from A, equivalently an ultrafilter in the Boolean algebra of definable sets over A.
- $S_n(A)$ is the set of all types in n free variables over A.
- T is stable if for some infinite cardinal λ, for every A, $|A| \leq \lambda$ implies $|S(A)| \leq \lambda$.
  $\lambda(T)$ is the least cardinal in which T is stable.
- A structure $\mathcal{M}$ is saturated if it realizes every type over every $A \subseteq M$ with $|A| < |M|$.
- $\{ \overline{a}_i : i \in I \}$ is a set of indiscernible sequences if for every permutation π of I and every (distinct) tuple $\langle i_1, \ldots, i_n \rangle$ from I, $\langle \overline{a}_{i_1}, \ldots, \overline{a}_{i_n} \rangle$ and $\left\langle \overline{a}_{\pi(i_1)}, \ldots, \overline{a}_{\pi(i_n)} \right\rangle$ have the same type.

“Forking calculus”

$$B \underset{A}{\downfree} C$$

“B is independent from C over A”

“The type of B over $C \cup A$ does not fork (d.n.f.) over A”

Summary from:
M. Makkai, A survey of basic stability theory, Israel J. Math, 1984.

“Forking calculus”

- T is a complete finitary first order theory which is stable.
- $\mathfrak{M}$ is a (very) large saturated model called the monster model or the universe.
- “small” means “a lot smaller” than the cardinality of $\mathfrak{M}$
- EVERYTHING(*) happens in $\mathfrak{M}$, in particular every set is a “small” subset of $\mathfrak{M}$.
- “automorphism” means “elementary automorphism of $\mathfrak{M}$”
- “algebraic” means “has only finitely many solutions”
- ...(*) Everything really happens in the “imaginary” universe $\mathfrak{M}^{\mathrm{eq}}$.

“Forking calculus”

First basic fact

Let p be a type over A.
Among the orbits by automorphisms fixing A of extensions of p to $\mathfrak{M}$, there is a unique orbit of ‘small’ size;
this is the orbit of non-forking ideal extensions of p.

The cardinality of this orbit is the multiplicity of p;
types of multiplicity one are stationary.

"Forking calculus"

First basic fact

Let p be a type over A.
Among the orbits by automorphisms fixing A of extensions of p to $\mathfrak{M}$, there is a unique orbit of 'small' size;
this is the orbit of non-forking ideal extensions of p.

The cardinality of this orbit is the multiplicity of p;
types of multiplicity one are stationary.

Second basic fact

Any ideal type is the unique non-forking extension to $\mathfrak{M}$ of its restriction to some set A, $|A| \leq |T|$.

INVARIANCE

α any automorphism of $\mathfrak{C}$

$$B \underset{A}{\downarrow} C \Rightarrow \alpha[B] \underset{\alpha[A]}{\downarrow} \alpha[C]$$

EXISTENCE

For all A, B, C:
there is α fixing A such that

$$\alpha[B] \underset{A}{\downarrow} C$$

TRANSITIVITY

$$A \subseteq B \subseteq C$$

$$D \underset{A}{\downarrow} B,\ D \underset{B}{\downarrow} C$$

$$\Longrightarrow D \underset{A}{\downarrow} C$$

SYMMETRY

$$B \underset{A}{\downarrow} C \longrightarrow C \underset{A}{\downarrow} B$$

MONOTONICITY

$$A \subseteq A' \subseteq A \cup C$$
$$C' \subseteq A \cup C$$

$$B \underset{A}{\downarrow} C \Longrightarrow B \underset{A'}{\downarrow} C'$$

CONTINUITY

$\langle A_i \rangle$, $\langle B_i \rangle$, $\langle C_i \rangle$, $(i \in D)$
up-directed families with unions A, B, C

$$B_i \underset{A_i}{\downarrow} C_i \text{ for all } i \in D \Longrightarrow B \underset{A}{\downarrow} C$$

OPEN MAPPING THEOREM

$$A \subseteq B$$
$$F = \{\, p \in S(B) :\ p \text{ d.n.f.}/a \,\}$$

$$F \longrightarrow S(A) : p \mapsto p \upharpoonright A$$

is an open, continuous, surjection.

FINITE CHARACTER

If $p \in S(B)$ forks over $A \subset B$,
then there is $\theta(\overline{x}, \overline{b}) \in p$ such that

$$\theta \in q \in S(A\overline{b}) \longrightarrow q \text{ forks over } A$$

SECOND BASIC FACT^{+}

There is a cardinal $\kappa(T) \leqslant |T|^{+}$,
which is the least infinite cardinal κ
such that for all $\overline{b}$, C,
there is $A \subseteq C$, $|A| < \kappa$, and

$$\overline{b} \underset{A}{\downarrow} C$$

SECOND BASIC FACT^{++}

For all B, C,
there is $A \subseteq C$, $|A| \leqslant \kappa(T) + |B|$, and

$$B \underset{A}{\downarrow} C$$

Orthogonality, Domination . . .

- orthogonality extends the independence relation from one between sets to one between types:
  (very) roughly speaking, if p and q are types over a suitably well-behaved set A, then p is orthogonal to q, $p \perp q$, if given any realizations b of p and c of q,

$$b \underset{A}{\downarrow} c$$

- Domination reflects how the properties of one type control the properties of another:
  B is dominated by C over A iff for any X,

$$X \underset{A}{\downarrow} C \Longrightarrow X \underset{A}{\downarrow} B$$

- Domination allows the transfer of suitable $\downarrow$ and $\perp$ relationships.

. . . weight, regular types, dimension

- The weight of B measures how big an independent set can depend on B; it is well-defined.

- $p \not\perp q_i$ $(i \in I)$ and $q_i \perp q_j$ for all $i \neq j \in I$ implies $|I| \leq \mathrm{wt}(p)$.

- The weight one sets/types are the sets for which $\mathop{\smile\!\!\!\!\!\!\!\;\mid}$ has a well-defined, useful, dimension function.

- The regular types are "minimal" weight one types:
  p is regular if it is stationary, non-algebraic, and for every forking extension q of p, $p \perp q$.

- $\not\perp$ is an equivalence relation on weight one types.

Algebra

Free algebras (and analogies)

- An algebra "freely generated" by an "independent basis"
- e.g. A vector space over a division ring and its basis.
- but also, *consider by analogy*:
  an algebraically closed field as the algebraic closure of a transcendence basis.
- The Fundamental Theorem of Finitely Generated Abelian Groups.

model theoretically interesting properties

Let $\mathcal{V}$ be a variety of algbras, X a set, and $\mathcal{F} = \mathcal{F}_{\mathcal{V}}(X)$.

- Any permutation of X extends to an automorphism of $\mathcal{F}$;
- Any (set) map $X \to \mathcal{F}$ extends to a homomorphism $\mathcal{F} \to \mathcal{F}$;
- If $\mathcal{A} \in \mathcal{V}$ and $|A| \leq |X|$, then $\mathcal{A}$ is a homomorphic image of $\mathcal{F}$.

There are lots of endomorphisms, lots of automorphisms, lots of homomorphisms.

model theoretically interesting properties

Let $\mathcal{V}$ be a variety of algbras, X a set, and $\mathcal{F} = \mathcal{F}_{\mathcal{V}}(X)$.

- Any permutation of X extends to an automorphism of $\mathcal{F}$;
- Any (set) map $X \to \mathcal{F}$ extends to a homomorphism $\mathcal{F} \to \mathcal{F}$;
- If $\mathcal{A} \in \mathcal{V}$ and $|A| \leq |X|$, then $\mathcal{A}$ is a homomorphic image of $\mathcal{F}$.

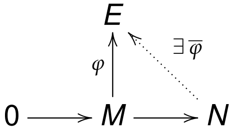
There are lots of endomorphisms, lots of automorphisms, lots of homomorphisms.

Freeness is a *projective* property:

it is characterized by maps *from* the free object.

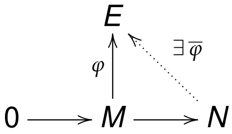
By contrast: injective modules

In an abelian category, but in particular in the category of all *R*-modules, an object *E* is injective if



By contrast: injective modules

In an abelian category, but in particular in the category of all R-modules, an object E is injective if



Equivalently

- E is a direct summand in every extension;
- Every formally consistent system of linear equations over E has a solution in E.

injective modules

- Every homomorphism from a submodule of an injective module E lifts to an endomorphism of E.

RICH endomorphism structure.

injective modules

- Every module M can be embedded into an injective module;
- ... in fact into a minimal injective extension called the injective envelope of M, $\mathrm{E}(M)$.

injective modules

- Every module M can be embedded into an injective module;
- ...in fact into a minimal injective extension called the injective envelope of M, $\mathrm{E}(M)$.
- (weak universality) Set D to be the direct sum of the injective envelopes of the simple modules. Then D is a cogenerator: every non zero module has a non zero homomorphism to D, and every module embeds in some power of D.

injective modules

- Every module M can be embedded into an injective module;
- ... in fact into a minimal injective extension called the injective envelope of M, $\mathrm{E}(M)$.
- (weak universality) Set D to be the direct sum of the injective envelopes of the simple modules. Then D is a cogenerator: every non zero module has a non zero homomorphism to D, and every module embeds in some power of D.

Injectivity is an example of an injective property:

it is characterized by maps into the object

Model Theory

Model Theory: some useful analogies

ALGEBRA	$\Longleftrightarrow$	MODEL THEORY
"generated by"	$\Longleftrightarrow$	• definable closure • algebraic closure
"algebraic independence"	$\Longleftrightarrow$	"non-forking independence" (of a suitable set)
"independence" "freeness" "non-structure"	$\Longleftrightarrow$	"indiscernible set"

Model Theory: some useful analogies

Surjections between structures are not very natural in model theory, but:

- Saturation is an injective property
  - universal + homogeneous
  - “Solves every consistent linear system” (in modules)
    $\sim$ realizes every type

Notation: Model theory

- T, a complete (stable) theory of cardinality τ.
- $\mathfrak{M}$, a saturated model of T of sufficiently large cardinality $\bar{\kappa}$.
- $\mathfrak{M}^{\mathrm{eq}}$, the imaginary universe, necessary especially for "definable" and "algebraic".
- (T superstable) There is a cardinal $\lambda(T) \leq 2^{\tau}$ such that T is stable in λ iff $\lambda \geq \lambda(T)$.
  T has a saturated model in every cardinal $\lambda \geq \lambda(T)$, and $\kappa(T) = \aleph_0$

strong type, $\mathtt{stp}(a/A)$, means the type of a over $\mathrm{acl}(A)$ (in $\mathfrak{M}^{\mathrm{eq}}$).
An *a*-model is one in which every strong type over a finite subset is realized.

Almost indiscernible theories

Almost indiscernible theories

Definition

Let T be a complete theory of cardinality τ, $\mu \leq \tau$ a finite or infinite cardinal, and $\kappa > \tau$ a cardinal.

T is called (μ, κ)-almost indiscernible if it has a saturated model M of cardinality κ which is in the algebraic closure of an indiscernible set I of μ-sequences.

T is almost indiscernible if it is (μ, κ)-almost indiscernible for some such μ, κ.

Almost indiscernible theories

Definition

Let T be a complete theory of cardinality τ, $\mu \leq \tau$ a finite or infinite cardinal, and $\kappa > \tau$ a cardinal.

T is called (μ, κ)-almost indiscernible if it has a saturated model M of cardinality κ which is in the algebraic closure of an indiscernible set I of μ-sequences.

T is almost indiscernible if it is (μ, κ)-almost indiscernible for some such μ, κ.

"Almost indiscernible" as defined by Pillay-Sklinos is "$(n, \aleph_1)$-almost indiscernible for some finite $n > 0$ and T countable".

Almost indiscernible theories: observations

- If T is (μ, κ)-almost indiscernible then it is (μ', κ)-almost indiscernible for any μ', $\mu \leq \mu' \leq \tau$,

- The example of the theory of the countable disjoint union of copies of $\langle \mathbb{Q}; \leq \rangle$ shows that $\kappa = \tau$ should be excluded from the Definition.

- The Definition does not require or even imply the existence of "best possible choices".

- We have an example of a countable $\aleph_1$-categorical theory which is not almost indiscernible.

Almost indiscernible theories: Context

1. T is a (μ, κ)-almost indiscernible theory, $|T| = \tau$, $\mu \leq \tau < \kappa$, with universe $\mathfrak{M}$ of some regular cardinality $\overline{\kappa} \gg \kappa$.
2. M is a saturated model as in the definition: $|M| = \kappa$, I is an indiscernible *set* of μ-sequences in M, and M is in the algebraic closure of the (union of) I.
   Since $\mu \leq \tau < \kappa$, necessarily $|I| = \kappa$, so we can write I as a κ-sequence $\langle \underline{\mathrm{e}}_\alpha : \alpha < \kappa \rangle$, and when necessary the μ-sequence $\underline{\mathrm{e}}_\alpha$ is indexed as $\langle \underline{\mathrm{e}}_{\alpha,i} : i < \mu \rangle$.
3. Extend I to an indiscernible 'set' $\bar{I} = \langle \underline{\mathrm{e}}_\alpha : \alpha < \overline{\kappa} \rangle$ in $\mathfrak{M}$.
   For each infinite *ordinal* $\lambda \leq \overline{\kappa}$, let $I_\lambda = \langle \underline{\mathrm{e}}_\alpha : \alpha < \lambda \rangle$ and set $M_\lambda = \mathrm{acl}(I_\lambda)$ in $\mathfrak{M}$.

Almost indiscernible theories: Basic facts: 1

Theorem (2.2)

λ denotes an infinite ordinal.

1. *$\lambda \geq \tau$ implies $|M_\lambda| = |\lambda|$ and $\lambda < \tau$ implies $\mu\, |\lambda| \leq |M_\lambda| \leq \tau$.*

Almost indiscernible theories: Basic facts: 1

Theorem (2.2)

λ denotes an infinite ordinal.

1. *$\lambda \geq \tau$ implies $|M_\lambda| = |\lambda|$ and $\lambda < \tau$ implies $\mu\,|\lambda| \leq |M_\lambda| \leq \tau$.*
2. *For all $\lambda \leq \overline{\kappa}$, $M_\lambda \preceq \mathfrak{M}$.*
   (And so $\langle M_\lambda \rangle_{\omega \leq \lambda \leq \overline{\kappa}}$ is an elementary chain.)

Theorem (2.2)

λ denotes an infinite ordinal.

1. *$\lambda \geq \tau$ implies $|M_\lambda| = |\lambda|$ and $\lambda < \tau$ implies $\mu\,|\lambda| \leq |M_\lambda| \leq \tau$.*
2. *For all $\lambda \leq \overline{\kappa}$, $M_\lambda \preceq \mathfrak{M}$.*
   (And so $\langle M_\lambda \rangle_{\omega \leq \lambda \leq \overline{\kappa}}$ is an elementary chain.)
3. *For all $\lambda \leq \overline{\kappa}$, M_λ is $|\lambda|$-saturated.*
4. *In particular $M_{\overline{\kappa}}$ is saturated of cardinality $\overline{\kappa}$, so without loss of generality equal to $\mathfrak{M}$.*

Almost indiscernible theories: Basic facts: 1

Theorem (2.2)

λ denotes an infinite ordinal.

1. *$\lambda \geq \tau$ implies $|M_\lambda| = |\lambda|$ and $\lambda < \tau$ implies $\mu\,|\lambda| \leq |M_\lambda| \leq \tau$.*
2. *For all $\lambda \leq \overline{\kappa}$, $M_\lambda \preceq \mathfrak{M}$.*
   (And so $\langle M_\lambda \rangle_{\omega \leq \lambda \leq \overline{\kappa}}$ is an elementary chain.)
3. *For all $\lambda \leq \overline{\kappa}$, M_λ is $|\lambda|$-saturated.*
4. *In particular $M_{\overline{\kappa}}$ is saturated of cardinality $\overline{\kappa}$, so without loss of generality equal to $\mathfrak{M}$.*
5. *For all $\lambda \leq \overline{\kappa}$, M_λ is $F^a_{|\lambda|}$-saturated.*

Almost indiscernible theories: Basic facts: 1

Theorem (2.2)

λ denotes an infinite ordinal.

1. *$\lambda \geq \tau$ implies $|M_\lambda| = |\lambda|$ and $\lambda < \tau$ implies $\mu\,|\lambda| \leq |M_\lambda| \leq \tau$.*
2. *For all $\lambda \leq \overline{\kappa}$, $M_\lambda \preceq \mathfrak{M}$.*
   (*And so $\langle M_\lambda \rangle_{\omega \leq \lambda \leq \overline{\kappa}}$ is an elementary chain.*)
3. *For all $\lambda \leq \overline{\kappa}$, M_λ is $|\lambda|$-saturated.*
4. *In particular $M_{\overline{\kappa}}$ is saturated of cardinality $\overline{\kappa}$, so without loss of generality equal to $\mathfrak{M}$.*
5. *For all $\lambda \leq \overline{\kappa}$, M_λ is $F^a_{|\lambda|}$-saturated.*
6. *In particular all M_λ have the property that all strong types over finite subsets are realized.*
   [*So it will follow from Theorem 2.4 to come, that since T is superstable, all the M_λ are a-models.*]

Almost indiscernible theories: Basic facts: 2

In particular, for cardinals $\nu \geq \tau$, $\langle M_\lambda : \nu \leq \lambda < \nu^+ \rangle$ is a elementary chain of copies of the saturated model of T of cardinality ν.

Corollary (2.3)

Let T be a complete theory of cardinality τ.

1. *Then T is (μ, κ)-almost indiscernible for some $\kappa > \tau$ iff T is (μ, τ^+)-almost indiscernible iff T is (μ, κ')-almost indiscernible for all $\kappa' > \tau$.*
2. *In particular, under the conditions of (1), M_τ is a saturated model which is the algebraic closure of an indiscernible sequence $\langle \underline{e}_\alpha : \alpha < \tau \rangle$ of μ-tuples.*

   So we can now assume without loss of generality that T is a complete theory of cardinality τ which is (μ, τ^+)-almost indiscernible for some $\mu \leq \tau$.

Almost indiscernible theories: Basic facts: 3

Theorem (2.4)

Let T be (μ, τ^+)*-almost indiscernible.*
Then T is stable in every cardinal $\lambda \geq \tau$ *hence:*

- *T is superstable, and*
- *in particular if T is countable and* $(\aleph_0, \aleph_1)$*-almost indiscernible, then T is totally transcendental.*

Almost indiscernible theories: Basic facts: 3

Theorem (2.4)

Let T be (μ, τ^+)-almost indiscernible.
Then T is stable in every cardinal $\lambda \geq \tau$ hence:

- *T is superstable, and*
- *in particular if T is countable and $(\aleph_0, \aleph_1)$-almost indiscernible, then T is totally transcendental.*

For uncountable T, all that follows in general for superstable theories is that T is stable in every cardinal $\geq 2^{|T|}$. So almost indiscernible theories are “strongly” superstable.
Any complete theory T in a (possibly uncountable language) which is categorical in $|T|^+$ is “strongly” superstable.

Almost indiscernible theories: Basic facts: 4

Definition

Set $\mathbf{p} = \mathbf{p}(\vec{v}) = \mathrm{tp}(\underline{e}_\omega / M_\omega) = \{\, \varphi(\vec{v}) : \ \models \varphi[\underline{e}_\omega] \,\}$.

Almost indiscernible theories: Basic facts: 4

Definition

Set $\mathbf{p} = \mathbf{p}(\vec{v}) = \mathrm{tp}(\underline{e}_\omega / M_\omega) = \{\, \varphi(\vec{v}) : \ \models \varphi[\underline{e}_\omega] \,\}$.

Proposition (2.6)

T is non-multidimensional.

Almost indiscernible theories: Structure: 1

Let $\overline{\mu}$ be $\aleph_0$ if μ is finite and μ^+ if μ is infinite.
Note that $\overline{\mu} \leq \tau^+$.

Almost indiscernible theories: Structure: 1

Let $\overline{\mu}$ be $\aleph_0$ if μ is finite and μ^+ if μ is infinite.
Note that $\overline{\mu} \leq \tau^+$.

Definition

We let $\mathcal{R}$ be the *set* of non-orthogonality equivalence classes of weight one types of T.
If p is some weight one type, then $[p]$ is its class.

Almost indiscernible theories: Structure: 2

Proposition (2.8)

Let $\lambda \geq \omega$ be an ordinal. [*Primarily $\lambda = \omega$ or $\lambda = \overline{\mu}$.*]
Consider $M_\lambda \prec M_{\lambda+1} = \mathrm{acl}(M_\lambda \cup \underline{e}_\lambda)$.
There is a set of elements $C = \{ c_j : j \in J \} \subset M_{\lambda+1} \setminus M_\lambda$, with J finite if μ is finite and $|J| \leq \mu$ otherwise, such that:

1. *C is independent over M_λ ,*
2. *each $\mathrm{tp}(c_j / M_\lambda)$ is regular,*
3. *all regular types occur up to non-orthogonality amongst the various types $\mathrm{tp}(c_j / M_\lambda)$,*

and such that $M_{\lambda+1}$ is a-prime and minimal over $M_\lambda \cup C$.
Without loss of generality, we can fix some set $\mathcal{Q}$ of regular types over M_λ representing the classes of $\mathcal{R}$ over M_λ , and assume that for each $c \in C$, $\mathrm{tp}(c / M_\lambda) \in \mathcal{Q}$, [so that for each c, $c' \in C$, either $\mathrm{tp}(c / M_\lambda) = \mathrm{tp}(c' / M_\lambda)$ or these types are orthogonal].

Proposition (2.9)

Continuing the notation of Proposition 2.8 (with $\lambda = \overline{\mu}$), there are μ-tuples $D = \{ d_j : j \in J \}$ such that:

1. $\mathtt{tp}(d_j / M_{\overline{\mu}})$ *has weight one and* $c_j \in \mathtt{acl}(M_{\overline{\mu}} \cup \{ d_j \})$ *for each* $j \in J$ *;*
2. *D is* $M_{\overline{\mu}}$*-independent; and*
3. $\underline{\mathrm{e}}_{\overline{\mu}}$ *is interalgebraic with D over* $M_{\overline{\mu}}$ *.*

[*Hence also the types of the tuples* d_j *represent all the classes of* $\mathcal{R}$ *over* $M_{\overline{\mu}}$ *.*]

Almost indiscernible theories: Structure: 4

Theorem (2.10)

Any model M which contains $M_{\overline{\mu}}$ is the algebraic closure of $M_{\overline{\mu}}$ together with an $M_{\overline{\mu}}$-independent set D of tuples of realizations of weight one types over $M_{\overline{\mu}}$.

Almost indiscernible theories: Structure: 4

Theorem (2.10)

Any model M which contains $M_{\overline{\mu}}$ is the algebraic closure of $M_{\overline{\mu}}$ together with an $M_{\overline{\mu}}$-independent set D of tuples of realizations of weight one types over $M_{\overline{\mu}}$.

Corollary (2.11)

We can find (in $M_{\overline{\mu}+1}$) sets $\{\, N(r) : r \in \mathcal{R} \,\}$, each uniquely determined up to isomorphism over $M_{\overline{\mu}}$ by r , with $\mathtt{tp}(N_r / M_{\overline{\mu}}) \in r$ and such that each $N(r)$ is a maximal (with respect to $\subseteq$) weight one set over $M_{\overline{\mu}}$.

We call $N(r)$ the hull of r (over $M_{\overline{\mu}}$).

Furthermore, if M is any model containing $M_{\overline{\mu}}$ then M is the algebraic closure of a family (independent over $M_{\overline{\mu}}$) of copies of the various $N(r)$, $r \in \mathcal{R}$.

Almost indiscernible theories: Structure: 5

This should all work over M_ω rather than over $M_{\overline{\mu}}$???

The particular case of theories of modules

T is a **complete superstable theory** of *R*-modules with $|T| = \tau$.

$\lambda(T)$ is the least cardinal in which *T* is stable (so $\tau \leq \lambda(T) \leq 2^{\tau}$).

T is a **complete superstable theory** of R-modules with $|T| = \tau$.

$\lambda(T)$ is the least cardinal in which T is stable (so $\tau \leq \lambda(T) \leq 2^{\tau}$).

So by Theorem 2.4, if T is almost indiscernible, then $\lambda(T) = \tau$.

Modules: Basic facts [Prest: Model theory of Modules]

- A positive primitive formula (ppf) is an existential quantification of a finite system of linear equations. The solution set of a ppf without parameters in a module N is a subgroup of N.

Modules: Basic facts [Prest: Model theory of Modules]

- A positive primitive formula (ppf) is an existential quantification of a finite system of linear equations. The solution set of a ppf without parameters in a module N is a subgroup of N.
- Modulo the theory of R-modules, every formula is equivalent to a boolean combination of ppfs.

Modules: Basic facts [Prest: Model theory of Modules]

- A positive primitive formula (ppf) is an existential quantification of a finite system of linear equations. The solution set of a ppf without parameters in a module N is a subgroup of N.
- Modulo the theory of R-modules, every formula is equivalent to a boolean combination of ppfs.
- Every complete theory of modules is stable.

Modules: Basic facts [Prest: Model theory of Modules]

- A positive primitive formula (ppf) is an existential quantification of a finite system of linear equations. The solution set of a ppf without parameters in a module N is a subgroup of N.
- Modulo the theory of R-modules, every formula is equivalent to a boolean combination of ppfs.
- Every complete theory of modules is stable.
- N is pure-injective (equationally compact) iff every system of linear equations over N [every system of pp formulas over N] which is finitely satisfiable in N is satisfied in N.

Modules: Basic facts [Prest: Model theory of Modules]

- A positive primitive formula (ppf) is an existential quantification of a finite system of linear equations. The solution set of a ppf without parameters in a module N is a subgroup of N.
- Modulo the theory of R-modules, every formula is equivalent to a boolean combination of ppfs.
- Every complete theory of modules is stable.
- N is pure-injective (equationally compact) iff every system of linear equations over N [every system of pp formulas over N] which is finitely satisfiable in N is satisfied in N.
- T is superstable iff there are no infinite descending chains of pp-definable subgroups, each of infinite index in the preceding.

Modules: Basic facts [Prest: Model theory of Modules]

- A positive primitive formula (ppf) is an existential quantification of a finite system of linear equations. The solution set of a ppf without parameters in a module N is a subgroup of N.
- Modulo the theory of R-modules, every formula is equivalent to a boolean combination of ppfs.
- Every complete theory of modules is stable.
- N is pure-injective (equationally compact) iff every system of linear equations over N [every system of pp formulas over N] which is finitely satisfiable in N is satisfied in N.
- T is superstable iff there are no infinite descending chains of pp-definable subgroups, each of infinite index in the preceding.
- T is tt iff there are no infinite descending chains of pp-definable subgroups.

Modules: Basic facts

Proposition

If M is superstable and $M \prec N \models T$, then the factor module N/M is totally transcendental.

tt modules have a very good structure theory.

Modules: Basic facts

Proposition

If M is superstable and $M \prec N \models T$, then the factor module N/M is totally transcendental.

tt modules have a very good structure theory.

Proposition

1. *$M \models T$ is an a-model iff M is pure-injective and weakly saturated.*

Modules: Basic facts

Proposition

If M is superstable and $M \prec N \models T$, then the factor module N/M is totally transcendental.

tt modules have a very good structure theory.

Proposition

1. *$M \models T$ is an a-model iff M is pure-injective and weakly saturated.*
2. *Pure-injective models of T are $\aleph_0$-homogeneous.*

Modules: Basic facts

Proposition

If M is superstable and $M \prec N \models T$, then the factor module N/M is totally transcendental.

tt modules have a very good structure theory.

Proposition

1. *$M \models T$ is an a-model iff M is pure-injective and weakly saturated.*
2. *Pure-injective models of T are $\aleph_0$-homogeneous.*
3. *Elementary extensions of pure-injective models are pure-injective.*

Modules: Basic facts

Proposition

If M is superstable and $M \prec N \models T$, then the factor module N/M is totally transcendental.

tt modules have a very good structure theory.

Proposition

1. *$M \models T$ is an a-model iff M is pure-injective and weakly saturated.*
2. *Pure-injective models of T are $\aleph_0$-homogeneous.*
3. *Elementary extensions of pure-injective models are pure-injective.*
4. *Elementary extensions of a-models are a-models.*

Modules: Basic facts

Proposition

If M is superstable and $M \prec N \models T$, then the factor module N/M is totally transcendental.

tt modules have a very good structure theory.

Proposition

1. *$M \models T$ is an a-model iff M is pure-injective and weakly saturated.*
2. *Pure-injective models of T are $\aleph_0$-homogeneous.*
3. *Elementary extensions of pure-injective models are pure-injective.*
4. *Elementary extensions of a-models are a-models.*

Proposition

In general, for any complete theory of modules T', if every a-model is pure-injective, then T' is superstable.

Superstable modules: Example

Let $R = N = \overline{\mathbb{Z}_{(p)}}$.

Let $T_0 = \mathrm{Th}({}_{\mathbb{Z}}N)$ and $T_1 = \mathrm{Th}({}_{R}N)$.

Superstable modules: Example

Let $R = N = \overline{\mathbb{Z}_{(p)}}$.

Let $T_0 = \mathrm{Th}({}_{\mathbb{Z}}N)$ and $T_1 = \mathrm{Th}({}_{R}N)$.

Then $|T_0| = \aleph_0$ and $|T_1| = 2^{\aleph_0}$.
In both cases $\lambda(T) = 2^{\aleph_0}$ and T is superstable not tt.

Superstable modules: Example

Let $R = N = \overline{\mathbb{Z}_{(p)}}$.

Let $T_0 = \mathrm{Th}({}_{\mathbb{Z}}N)$ and $T_1 = \mathrm{Th}({}_{R}N)$.

Then $|T_0| = \aleph_0$ and $|T_1| = 2^{\aleph_0}$.
In both cases $\lambda(T) = 2^{\aleph_0}$ and T is superstable not tt.

T_0 is not almost indiscernible.

T_1 is $(2^{\aleph_0}, (2^{\aleph_0})^+)$-almost indiscernible.

Almost indiscernible modules

Remember that an almost indiscernible theory must be superstable.

Almost indiscernible modules

Remember that an almost indiscernible theory must be superstable.

Theorem (2.17)

Let T be a superstable theory of modules with $\lambda(T) = |T| = \tau$, (in particular, if T is totally transcendental).
Then T is an almost indiscernible theory of modules.

Almost indiscernible modules

Remember that an almost indiscernible theory must be superstable.

Theorem (2.17)

Let T be a superstable theory of modules with $\lambda(T) = |T| = \tau$, (in particular, if T is totally transcendental).
Then T is an almost indiscernible theory of modules.

Corollary (2.18)

A complete theory T of modules is almost indiscernible
iff
it is superstable with $\lambda(T) = |T|$.

Saturated Free Algebras

Saturated Free Algebras

We provide generalizations and extensions of the results in Pillay and Sklinos to the uncountable context.

Let $\mathcal{V}$ be a variety over an algebraic language $\mathcal{L}$ of cardinality $\tau \geq \aleph_0$.

Let the algebra *M* be a free algebra for $\mathcal{V}$ on a set

$$I = \{ \underline{e}_\alpha : \alpha < \tau^+ \}$$

(of individual elements), such that *M* is τ^+ saturated.

T always means $\mathrm{Th}(M)$.

Saturated Free Algebras: Results of Pillay-Sklinos

Definition

We call $B \subset M$ *basic* if it is a subset of some free basis of M.
We call $b \in M$ *basic* if $\{ b \}$ is basic.

Definition

We call $B \subset M$ *basic* if it is a subset of some free basis of M.
We call $b \in M$ *basic* if $\{ b \}$ is basic.

Lemma (3.7)

There is a type p_0 over $\emptyset$ such that for any $a \in M$,
a is basic iff a realizes p_0.

Saturated Free Algebras: Results of Pillay-Sklinos

Definition

We call $B \subset M$ *basic* if it is a subset of some free basis of M.
We call $b \in M$ *basic* if $\{ b \}$ is basic.

Lemma (3.7)

There is a type p_0 over $\emptyset$ such that for any $a \in M$,
a is basic iff a realizes p_0.

Lemma (3.8)

The type p_0 is stationary.
Hence so is ${p_0}^{(n)}$ for any n.

Saturated free algebras: Results of Pillay-Sklinos

Proposition (3.11)

The sequence I is a Morley sequence in p_0 .

Saturated free algebras: Results of Pillay-Sklinos

Proposition (3.11)

The sequence I is a Morley sequence in p_0 .

Proposition (3.12)

Let $\overline{a}$, $\overline{b} \in M$.
Then $\overline{a}$ is independent from $\overline{b}$ over $\emptyset$ iff there is a basis $A \cup B$, A, B disjoint, of M such that $\overline{a} \in \langle\langle A \rangle\rangle$ and $\overline{b} \in \langle\langle B \rangle\rangle$.

Proposition (3.11)

The sequence I is a Morley sequence in p_0.

Proposition (3.12)

Let $\overline{a}, \overline{b} \in M$.
Then $\overline{a}$ is independent from $\overline{b}$ over $\emptyset$ iff there is a basis $A \cup B$, A, B disjoint, of M such that $\overline{a} \in \langle\langle A \rangle\rangle$ and $\overline{b} \in \langle\langle B \rangle\rangle$.

The proof of 3.12 extends in the obvious ways to independence of infinite tuples, and independence over an arbitrary basic set.

Saturated free modules

$\mathcal{V}$ is a variety of modules over a ring R such that a sufficiently large free module M of $\mathcal{V}$ is saturated; $T = \mathrm{Th}(M)$.

The pp parts of 1-types over $\emptyset$ are ordered by

$$p \leq q \text{ iff for all } M \models T,\ p[M] \subseteq q[M]$$

“maximal” means maximal with respect to this order.

${p_0}^+$ is the pp-part of the type p_0 of a basic element.

Corollary (3.5(b))

T is totally transcendental, and $T = \mathrm{Th}(\mathcal{F}^{(\aleph_0)})$, $\mathcal{F}$ the free module in V on one generator.

Proposition (3.14)

The pp type $p_0{}^{+}$ of a basic element is maximal, hence p_0 has maximal Morley rank.

Saturated free modules

Proposition (3.14)

The pp type ${p_0}^+$ of a basic element is maximal, hence p_0 has maximal Morley rank.

Pillay and Sklinos ask whether the theory of a saturated free algebra must have finite Morley rank.

We provide an example showing that this is not true in general even for theories of modules.

Saturated free modules

Theorem (3.15)

Given a ring R of cardinality τ *,* $R^{(\tau^+)}$ (*regarded as a left R-module*) *is* τ^+*-saturated if and only if R is left perfect and right coherent.*

Saturated free modules

Theorem (3.15)

Given a ring R of cardinality τ, $R^{(\tau^+)}$ (regarded as a left R-module) is τ^+-saturated if and only if R is left perfect and right coherent.

Let $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q}(x) \\ 0 & \mathbb{Q}(x) \end{pmatrix}$.

The free right module on $\aleph_1$ generators is saturated.

p_0 has rank $\omega + 1$.

Questions and open problems

1. Is there a fundamental difference between theories that are (τ, τ^+) almost indiscernible and those that are (μ, τ^+) almost indiscernible for some $\mu < \tau$?

Questions and open problems

1. Is there a fundamental difference between theories that are (τ, τ^+) almost indiscernible and those that are (μ, τ^+) almost indiscernible for some $\mu < \tau$?

   We had results giving characterizations of algebraic closure [3.10] and independence in a saturated free algebra [3.12].
2. Are there similar results for arbitrary almost indiscernible theories?
3. Is there a more general description of independence in the theory of a saturated free algebra?

   For instance, something along the lines of:
   $\overline{a}$ and $\overline{b}$ are independent over $\overline{c}$ iff there is a basis X, the disjoint union of A, B, and C, such that $\overline{c} \in \langle\langle C \rangle\rangle$ and $\overline{a} \in \langle\langle A \cup C \rangle\rangle$, $\overline{b} \in \langle\langle B \cup C \rangle\rangle$.

Questions and open problems

4. Is there any kind of classification of those varieties $\mathcal{V}$ for which the free algebra on τ^+ generators is τ^+-saturated?

Questions and open problems

4. Is there any kind of classification of those varieties $\mathcal{V}$ for which the free algebra on τ^+ generators is τ^+-saturated?

5. What about Question 4, assuming the stability of $\mathcal{V}$, that is, that every completion of $\mathtt{Th}(\mathcal{V})$ is stable?

Questions and open problems

4. Is there any kind of classification of those varieties $\mathcal{V}$ for which the free algebra on τ^+ generators is τ^+-saturated?

5. What about Question 4, assuming the stability of $\mathcal{V}$, that is, that every completion of $\mathtt{Th}(\mathcal{V})$ is stable?

6. In Proposition [3.14] we showed that the rank of the type of a basic element of a large saturated free module is maximal.

   Is this true for large saturated free algebras in general?

Questions and open problems

For the rest, let us assume that the free algebra M on τ^+-generators is τ^+ saturated, and let $T = \mathtt{Th}(M)$.

7. Is T totally transcendental?

Questions and open problems

For the rest, let us assume that the free algebra M on τ^+-generators is τ^+ saturated, and let $T = \mathrm{Th}(M)$.

7. Is T totally transcendental?

8. Is there a structure theorem for the algebra M, for example as some kind of a product of a module and of a combinatorial part, along the lines of McKenzie and Valeriote [1989]?

Questions and open problems

Implicit (in at least) (7) and (8) is a rather vague:

9. Is there some kind of relative quantifier elimination theorem for such theories?

Questions and open problems

Implicit (in at least) (7) and (8) is a rather vague:

9. Is there some kind of relative quantifier elimination theorem for such theories?

and finally,

10. Investigate almost indiscernible theories in other classes of algebas, such as groups.

Acknowledgements

- Anand Pillay [NSF grants DMS-1665035 and 1760413],
- the University of Notre Dame,
- the University of Manitoba,
- the University of Manchester,
- Mike Prest,
- the very careful and attentive referees at Algebra Universalis,
- and the many people who have patiently listened to us talk about this material in various forms since late 2017.