Solution sets and polymorphism-homogeneity

Tamás Waldhauser
joint work with Endre Tóth
University of Szeged

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Basic examples

- A set $S \subseteq K^n$ of tuples over a field K is the solution set of a system of homogeneous linear equations if and only if S is a subspace, i.e., S is closed under linear combinations.
- A set $S \subseteq K^n$ of tuples over a field K is the solution set of a system of arbitrary linear equations if and only if S is an affine subspace, i.e., S is closed under affine combinations.

Questions

1. Can you always characterize solution sets of systems of equations as sets of tuples that are closed under something?

Answer: Sometimes.

2. What should be this something?

Answer: It must be the centralizer.

Setup

- A is a finite set.
- $\mathcal{O}_A$ is the set of all operations on A.
- $\mathbb{A} = (A, F)$ is an algebra ($F \subseteq \mathcal{O}_A$).
- $C = \operatorname{Clo} \mathbb{A} = \operatorname{Clo} F$ is the clone of term operations of $\mathbb{A}$.
  - A clone is a set of operations that is closed under compositions and contains the projections.
- An equation over $\mathbb{A}$ (or over C) is a pair (f, g), where $f, g \in \mathcal{O}_A^{(n)}$; the solution set of this equation is

$$\operatorname{Sol}(f, g) = \{\, \underline{a} \in A^n : f(\underline{a}) = g(\underline{a}) \,\}.$$

- A subset $S \subseteq A^n$ is an algebraic set if it is the solution set of a system of equations, i.e., if S is the intersection of (finitely many) sets of the form $\operatorname{Sol}(f, g)$.
- The collection of all algebraic sets over $\mathbb{A}$ is called the algebraic geometry of $\mathbb{A}$. (Plotkin, ∼1995 and Pinus, ∼2009)

Relations

- $\mathcal{R}_A$ denotes the set of all relations on A.
- For $R \subseteq \mathcal{R}_A$, a primitive positive formula $\Phi(x_1, \ldots, x_n)$ over R is an existentially quantified conjunction:
$$\Phi(x_1, \ldots, x_n) = \exists y_1 \cdots \exists y_m \mathop{\&}_{i=1}^{t} \rho_i\big(z_1^{(i)}, \ldots, z_{r_i}^{(i)}\big),$$
where $\rho_i \in R$ and $z_j^{(i)} \in \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$.
- The set of all relations definable by primitive positive formulas over R is denoted by $\langle R \rangle_\exists$. If $\langle R \rangle_\exists = R$, then we say that R is a relational clone.
- The set of all relations definable by quantifier-free primitive positive formulas over R is denoted by $\langle R \rangle_\nexists$. If $\langle R \rangle_\nexists = R$, then we say that R is a weak relational clone.
- An operation $f \in \mathcal{O}_A^{(n)}$ preserves a relation $\rho \subseteq A^h$ if ρ is a subalgebra of $(A, f)^h$. Notation: $f \rhd \rho$.

Galois connection between operations and relations

$$\mathrm{Pol}\, R = \{f \in \mathcal{O}_A : f \rhd \rho \text{ for all } \rho \in R\}$$

$$\mathrm{Inv}\, F = \{\rho \in \mathcal{O}_A : f \rhd \rho \text{ for all } f \in F\}$$

Theorem (Bodnarčuk, Kalužnin, Kotov, Romov, 1969 and Geiger, 1968)

For all $R \subseteq \mathcal{R}_A$ and $F \subseteq \mathcal{O}_A$, we have

$$\mathrm{Inv}\,\mathrm{Pol}\, R = \langle R \rangle_\exists \quad \text{and} \quad \mathrm{Pol}\,\mathrm{Inv}\, F = \mathrm{Clo}\, F.$$

The graph of an operation

The graph of $f \in \mathcal{O}_A^{(n)}$ is the following $(n+1)$-ary relation:

$$f^\bullet = \{(a_1, \dots, a_{n+1}) : f(a_1, \dots, a_n) = a_{n+1}\} \subseteq A^{n+1}.$$

For $F \subseteq \mathcal{O}_A$, let $F^\bullet = \{f^\bullet : f \in F\}$.

Commutation

The operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ commute (notation: $f \perp g$) if the following equivalent conditions hold:

- $f \rhd g^\bullet$;
- $g \rhd f^\bullet$;
- $f\colon (A, g)^n \to (A, g)$ is a homomorphism;
- $g\colon (A, f)^m \to (A, f)$ is a homomorphism.

Centralizer

The centralizer of $F \subseteq \mathcal{O}_A$ is the clone (called a primitive positive clone)

$$F^* = \{g \in \mathcal{O}_A : f \perp g \text{ for all } f \in F\} = \operatorname{Pol} F^\bullet.$$

Theorem (Burris, Willard, 1987)

If A is finite, then there are finitely many primitive positive clones on A.

An analogue of $F^\bullet$

For $F \subseteq \mathcal{O}_A$, let $F^\circ = \{\,\mathrm{Sol}(f,g) : n \in \mathbb{N} \text{ and } f,g \in F^{(n)}\}$.

Simple observations

For any finite algebra $\mathbb{A} = (A, F)$ with $\mathrm{Clo}\, F = C$, we have

1. $C^\bullet \subseteq C^\circ$
2. $\langle C^\bullet \rangle_\exists = \langle C^\circ \rangle_\exists$
3. $S \subseteq A^n$ is an algebraic set if and only if $S \in \langle C^\circ \rangle_\nexists$
4. $\mathrm{Inv}\, C^* = \langle C^\bullet \rangle_\exists$

Corollary

- Algebraic sets are closed under the centralizer C^*.
- The converse holds iff $\langle C^\circ \rangle_\exists = \langle C^\circ \rangle_\nexists$, i.e., if $\langle C^\circ \rangle_\nexists$ is a relational clone.

Simple observations

1. $C^\bullet \subseteq C^\circ$
2. $\langle C^\bullet \rangle_\exists = \langle C^\circ \rangle_\exists$
3. $S \subseteq A^n$ is an algebraic set if and only if $S \in \langle C^\circ \rangle_{\nexists}$
4. $\operatorname{Inv} C^* = \langle C^\bullet \rangle_\exists$

Only the centralizer

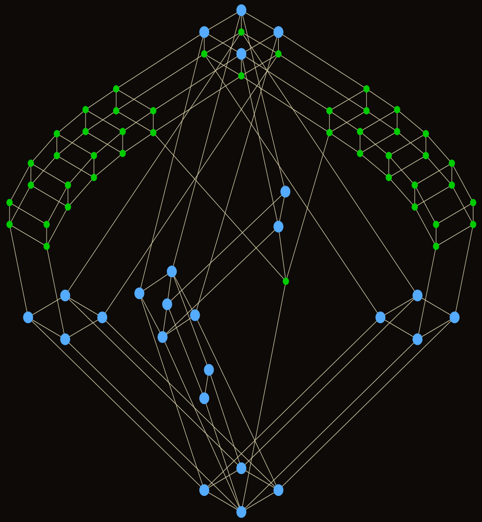
If there is a clone D such that the algebraic sets are exactly the D-closed sets, then $D = C^*$.

Name that kid

If the above holds, i.e., if solution sets of systems of equations over $\mathbb{A}$ are exactly the centralizer-closed sets, then we say that the algebra $\mathbb{A}$ (or the clone C) has property (SDC).

Theorem (Post, 1920)

There are countably infinitely many clones of Boolean functions:



Theorem

Every two-element algebra has property (SDC).

Corollary

There are 25 algebraic geometries on the two-element set.

Partial operations

- An n-ary partial operation on A is a map $h \colon \operatorname{dom} h \to A$ with $\operatorname{dom} h \subseteq A^n$.
- The set of all partial operations on A is denoted by $\mathcal{P}_A$.
- Preservation of relations can be defined for partial operations, and this induces the pPol – Inv Galois connection:

$$\operatorname{pPol} R = \{h \in \mathcal{P}_A : h \rhd \rho \text{ for all } \rho \in R\}$$

$$\operatorname{Inv} F = \{\rho \in \mathcal{O}_A : h \rhd \rho \text{ for all } h \in F\}$$

- A set $C \subseteq \mathcal{P}_A$ of partial operations is a strong partial clone if
  - C is closed under composition;
  - C contains the projections;
  - C is closed under restrictions.

Theorem (Romov, 1981)

For all $R \subseteq \mathcal{R}_A$ and $F \subseteq \mathcal{P}_A$, we have

$$\operatorname{Inv} \operatorname{pPol} R = \langle R \rangle_{\nexists} \quad \text{and} \quad \operatorname{pPol} \operatorname{Inv} F = \operatorname{Str} F.$$

Homogeneity

- A first-order structure $\mathcal{A}$ is said to be homomorphism-homogeneous, if every homomorphism $h\colon \mathcal{B} \to \mathcal{A}$ defined on a finitely generated substructure $\mathcal{B} \leq \mathcal{A}$ extends to a homomorphism $\widehat{h}\colon \mathcal{A} \to \mathcal{A}$ (Cameron, Nešetřil, 2006).
- A first-order structure $\mathcal{A}$ is said to be polymorphism-homogeneous, if every homomorphism $h\colon \mathcal{B} \to \mathcal{A}$ defined on a finitely generated substructure $\mathcal{B} \leq \mathcal{A}^k$ extends to a homomorphism $\widehat{h}\colon \mathcal{A}^k \to \mathcal{A}$ (Pech, Pech, 2015).

Theorem (Pech, Pech, 2015)

A finite relational structure has quantifier elimination for primitive positive formulas if and only if it is polymorphism-homogeneous.

Corollary

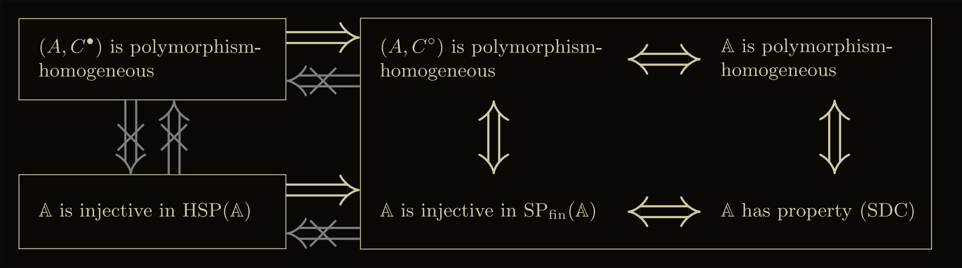
If $\mathbb{A}$ is a finite algebra and $C = \operatorname{Clo} \mathbb{A}$, then $\mathbb{A}$ has property (SDC) if and only if the relational structure (A, C°) is polymorphism-homogeneous.

Injectivity

We say that an algebra $\mathbb{A}$ is injective in a class of algebras $\mathcal{K}$, if every homomorphism $h\colon \mathbb{B} \to \mathbb{A}$ extends to a homomorphism $\widehat{h}\colon \mathbb{C} \to \mathbb{A}$ whenever $\mathbb{B}, \mathbb{C} \in \mathcal{K}$ and $\mathbb{B} \leq \mathbb{C}$.

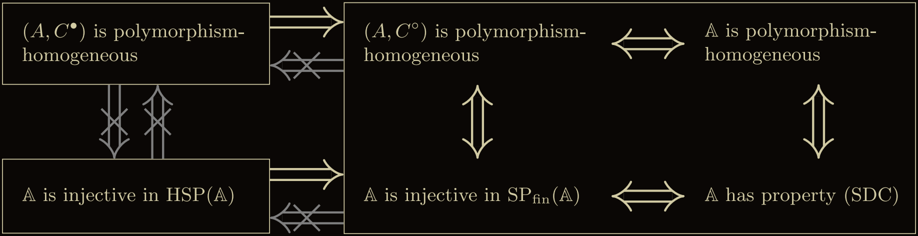
Theorem

A finite algebra $\mathbb{A}$ is polymorphism-homogeneous if and only if $\mathbb{A}$ is injective in $\mathsf{SP}_{\mathsf{fin}}(\mathbb{A})$.



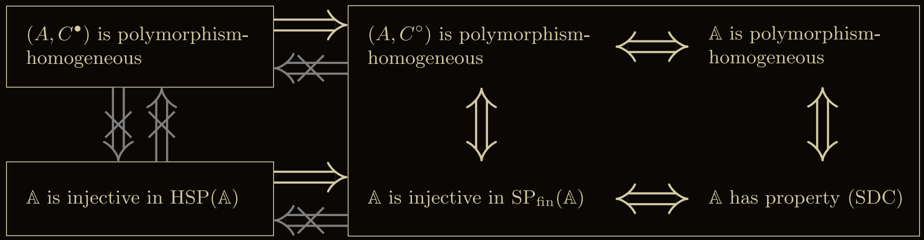
Finite abelian groups

- The following are equivalent:
  - $\mathbb{A}$ has property (SDC);
  - $\mathbb{A}$ is pol-hom;
  - (A, C°) is pol-hom;
  - $(A, C^\bullet)$ is pol-hom;
  - $\mathbb{A}$ is injective in $\mathrm{SP}_{\mathrm{fin}}\,\mathbb{A}$;
  - $\mathbb{A}$ is injective in $\mathrm{HSP}\,\mathbb{A}$;
  - the Sylow subgroups of $\mathbb{A}$ are homocyclic.



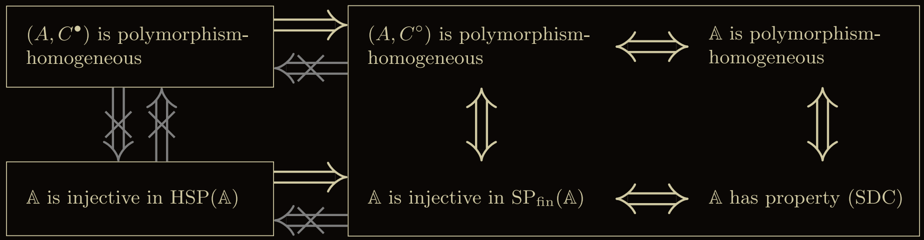
Finite semilattices

- The following are equivalent:
  - $\mathbb{A}$ has property (SDC);
  - $\mathbb{A}$ is pol-hom;
  - (A, C°) is pol-hom;
  - $\mathbb{A}$ is injective in $\mathrm{SP}_{\mathrm{fin}}\, \mathbb{A}$;
  - $\mathbb{A}$ is injective in $\mathrm{HSP}\, \mathbb{A}$ (Bruns, Lakser 1970 and Horn, Kimura 1971);
  - $\mathbb{A}$ is distributive.
- The following are also equivalent:
  - $(A, C^\bullet)$ is pol-hom;
  - $|A| = 1$.



Finite lattices

- The following are equivalent:
  - → $\mathbb{A}$ has property (SDC);
  - → $\mathbb{A}$ is pol-hom (Dolinka, Mašulović, 2011);
  - → (A, C°) is pol-hom;
  - → $\mathbb{A}$ is injective in $\mathsf{SP}_{\mathsf{fin}}\,\mathbb{A}$;
  - → $\mathbb{A}$ is injective in $\mathsf{HSP}\,\mathbb{A}$ (Balbes 1967);
  - → $\mathbb{A}$ is a finite Boolean lattice.
- The following are also equivalent:
  - → $(A, C^\bullet)$ is pol-hom;
  - → $|A| = 1$.



Finite monounary algebras $\mathbb{A} = (A, f)$

- The following are equivalent:
  - $\mathbb{A}$ has property (SDC);
  - $\mathbb{A}$ is pol-hom (Farkasová, Jakubíková-Studenovská, 2015);
  - (A, C°) is pol-hom;
  - $\mathbb{A}$ is injective in $\mathrm{SP}_{\mathrm{fin}}\,\mathbb{A}$;
  - all sources of $\mathbb{A}$ have the same height (or there are no sources).
- The following are also equivalent:
  - $(A, C^{\bullet})$ is pol-hom;
  - f is either bijective or constant.
- And the following are also equivalent:
  - $\mathbb{A}$ is injective in $\mathrm{HSP}\,\mathbb{A}$;
  - all sources of $\mathbb{A}$ have the same height and f has a fixed point (Czédli, Jakubíková-Studenovská, < 2000).

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The results presented in this talk

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- E. Tóth and T. Waldhauser, *Solution sets of systems of equations over finite lattices and semilattices*, Algebra Universalis **81** (2020), no. 2, 13.
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To do

- Characterize property (SDC) in your favorite class of algebras.
- Prove that all primitive positive clones have property (SDC).
- Determine the number of algebraic geometries on a finite set.
- Prove something about property (SDC) for clones on the three-element set.