EXPANSIONS OF ABELIAN SQUAREFREE GROUPS

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History

Definition

Let $C \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n}$. We call C a **clonoid** with source A and target algebra $\mathbf{B}$ if

(1) for all $k \in \mathbb{N}$: $C^{[k]} = C \cap B^{A^k}$ is a subuniverse of $\mathbf{B}^{A^k}$, and

(2) for all $k, n \in \mathbb{N}$, for all $(i_1, \dots, i_k) \in \{1, \dots, n\}^k$, and for all $c \in C^{[k]}$, the function $c' : A^n \to B$ with $c'(a_1, \dots, a_n) := c(a_{i_1}, \dots, a_{i_k})$ lies in $C^{[n]}$.

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E. Aichinger and P. Mayr. Finitely generated equational classes. *J. Pure Appl. Algebra*, 2016.

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E. Aichinger and P. Mayr. Finitely generated equational classes. *J. Pure Appl. Algebra*, 2016.

J. Bulín, A. Krokhin, and J. Opršal. Algebraic approach to promise constraint satisfaction. *Proceedings of the Annual ACM Symposium on Theory of Computing*, 2019.

Clonoid theory has been used to give an algebraic approach to PCSPs

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Let A be a nonempty set. A closed set of operations (**clone**) on A is a set of operations on A such that contains all projections and is closed under composition of functions.

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Let A be a nonempty set. Clearly the set of all operations on A is a clone.

Definition

Let $\mathbf{A}$ be an algebra. We call the smallest clone that contains the fundamental operations of $\mathbf{A}$ **term clone** or **clone** of $\mathbf{A}$. We call the smallest clone that contains the fundamental operations of $\mathbf{A}$ and the constant unary functions **polynomial clone** of $\mathbf{A}$.

History

E. L. Post. The two-valued iterative systems of mathematical logic. *Annals of Mathematics Studies*, 1941. (Description of the lattice of all clones on a two-element set).

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E. Aichinger, P. Mayr, and R. McKenzie. On the number of finite algebraic structures *Journal of the European Mathematical Society.*, 2014. (At most countably many Mal'cev clones on a finite set A).

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A. Krokhin, A. A. Bulatov, and P. Jeavons. The complexity of constraint satisfaction: an algebraic approach. *Structural theory of automata, semigroups, and universal algebra*, 2005. (Link between clones and CSPs).

$(\mathbb{F}, \mathbb{K})$-linearly closed clonoids

Main achievements:

We show that these clonoids are finitely many;

We provide a characterization of the lattice of these clonoids;

We provide an upper bound for the number of these clonoids.

$(\mathbb{F}, \mathbb{K})$-linearly closed clonoids

Definition

Let $m, s \in \mathbb{N}$, let $q_1, \dots, q_m, p_1, \dots p_s$ be powers of different primes, and let $\mathbb{K} = \prod_{i=1}^{m} \mathbb{F}_{q_i}$, $\mathbb{F} = \prod_{i=1}^{s} \mathbb{F}_{p_i}$ be products of fields of orders $q_1, \dots, q_m, p_1, \dots p_s$. An $(\mathbb{F}, \mathbb{K})$**-linearly closed clonoid** is a non-empty subset C of $\bigcup_{k \in \mathbb{N}} \prod_{i=1}^{s} \mathbb{F}_{p_i}^{\prod_{j=1}^{m} \mathbb{F}_{q_j}^k}$ with the following properties:

(1) for all $n \in \mathbb{N}$, $\boldsymbol{a}, \boldsymbol{b} \in \prod_{i=1}^{s} \mathbb{F}_{p_i}$, and $f, g \in C^{[n]}$:

$$\boldsymbol{a} f + \boldsymbol{b} g \in C^{[n]};$$

(2) for all $l, n \in \mathbb{N}$, $f \in C^{[n]}$, $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \in \prod_{j=1}^{m} \mathbb{F}_{q_j}^{l}$, and $A_i \in \mathbb{F}_{q_i}^{n \times l}$:

$$g : (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \mapsto f(A_1 \cdot \boldsymbol{x}_1^t, \cdots, A_m \cdot \boldsymbol{x}_m^t) \text{ is in } C^{[l]},$$

where with the juxtaposition $\boldsymbol{a} f$ we denote the Hadamard product of the two vectors (i.e. the component-wise product $(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n)$).

Theorem SF

Let $q_1, \ldots, q_m, p_1, \ldots p_s$ be powers of different primes, and let $\mathbb{K} = \prod_{i=1}^{m} \mathbb{F}_{q_i}$, $\mathbb{F} = \prod_{i=1}^{s} \mathbb{F}_{p_i}$ be products of fields of orders $q_1, \ldots, q_m, p_1, \ldots p_s$. Then every $(\mathbb{F}, \mathbb{K})$-linearly closed clonoid is generated by a set of unary functions.

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Theorem SF

Let $\mathbb{F} = \prod_{i=1}^{s} \mathbb{F}_{p_i}$ and $\mathbb{K} = \prod_{i=1}^{m} \mathbb{F}_{q_i}$ be products of finite fields of pair-wise co-prime order. Then there are finitely many distinct $(\mathbb{F}, \mathbb{K})$-linearly closed clonoids.

A characterization

Definition

Let $\langle M, +\rangle$ be a monoid and let $\langle R, +, \odot\rangle$ be a commutative ring with identity. Let

$$S := \{f \in R^M \mid f(a) \neq 0 \text{ for only finitely many } a \in M\}.$$

We define the **monoid ring** of M over R as the ring $(S, +, \cdot)$, where $+$ is the point-wise addition of functions and $(\sigma \cdot \rho)(a) := \sum_{b\in M} \sigma(b) \odot \rho(a - b)$. We denote by $R[M]$ the monoid ring of M over R.

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Definition

For a set I and $R \subseteq A^I$, $S \subseteq B^I$ let $\mathrm{Pol}(R, S) := \{f : A^k \to B : k \in \mathbb{N}, f(R, \ldots, R) \subseteq S\}$ denote the set of finitary functions preserving (R, S). We call $\mathrm{Pol}(R, S)$ the set of polymorphisms of the relational pair (R, S).

A characterization

Lemma

Let $p, q_1, \ldots, q_m$ be powers of distinct primes. Let U be the $(\prod_{i=1}^m \mathbb{F}_{q_i}, \cdot)$-submodule of $\prod_{i=1}^m \mathbb{F}_{q_i}^{\prod_{i=1}^m \mathbb{F}_{q_i}}$ that is generated by the identity map on $\prod_{i=1}^m \mathbb{F}_{q_i}$, and let V be an $(\mathbb{F}_p[\prod_{i=1}^m \mathbb{F}_{q_i}^\times], *)$-submodule of $\mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}}$. Then $\mathrm{Pol}(U, V)$ is an $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoid with unary part V.

A characterization

Lemma

Let $p, q_1, \dots, q_m$ be powers of distinct primes. Let U be the $(\prod_{i=1}^m \mathbb{F}_{q_i}, \cdot)$-submodule of $\prod_{i=1}^m \mathbb{F}_{q_i}^{\prod_{i=1}^m \mathbb{F}_{q_i}}$ that is generated by the identity map on $\prod_{i=1}^m \mathbb{F}_{q_i}$, and let V be an $(\mathbb{F}_p[\prod_{i=1}^m \mathbb{F}_{q_i}^\times], *)$-submodule of $\mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}}$. Then $\mathrm{Pol}(U, V)$ is an $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoid with unary part V.

Lemma

Let $p, q_1, \dots, q_m$ be powers of distinct primes. Then every $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoid is finitely related.

A characterization

Definition

Let $\mathbb{F}_p$ and $\mathbb{F}_{q_1}, \dots \mathbb{F}_{q_m}$ be finite fields and let $\mathbb{F}_{q_i}^{\times} = (\mathbb{F}_{q_i}, \cdot)$ be the multiplicative monoid reduct of $\mathbb{F}_{q_i}$, for all $i \in [m]$. We define the action $* : \mathbb{F}_p[\prod_{i=1}^m \mathbb{F}_{q_i}^{\times}] \times \mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}} \to \mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}}$ for all $\boldsymbol{a} \in \prod_{i=1}^m \mathbb{F}_{q_i}^{\times}$ and $f \in \mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}}$ by

$$(\tau_{\boldsymbol{a}} * f)(\boldsymbol{x}) = f(a_1 x_1, \dots, a_n x_n).$$

So for $\sigma \in \mathbb{F}_p[\prod_{i=1}^m \mathbb{F}_{q_i}^{\times}]$ with $\rho = \sum_{\boldsymbol{a} \in \prod_{i=1}^m \mathbb{F}_{q_i}^{\times}} z_{\boldsymbol{a}} \tau_{\boldsymbol{a}}$, then

$$(\sigma * f)(\boldsymbol{x}) = \sum_{\boldsymbol{a} \in \prod_{i=1}^m \mathbb{F}_{q_i}^{\times}} z_{\boldsymbol{a}} f(a_1 x_1, \dots, a_n x_n).$$

Corollary

Let $p, q_1, \ldots, q_m$ be powers of distinct primes. Then the function $\pi^{[1]}$ that sends an $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoid to its unary part is an isomorphism between the lattice of all $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoids and the lattice of all $(\mathbb{F}_p[\prod_{i=1}^m \mathbb{F}_{q_i}^\times], *)$-submodules of $\mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}}$.

Corollary

Let $p, q_1, \ldots, q_m$ be powers of distinct primes. Then the function $\pi^{[1]}$ that sends an $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoid to its unary part is an isomorphism between the lattice of all $(\mathbb{F}_p, \prod_{i=1}^m \mathbb{F}_{q_i})$-linearly closed clonoids and the lattice of all $(\mathbb{F}_p[\prod_{i=1}^m \mathbb{F}_{q_i}^\times], *)$-submodules of $\mathbb{F}_p^{\prod_{i=1}^m \mathbb{F}_{q_i}}$.

Theorem

Let $\mathbb{F} = \prod_{i=1}^s \mathbb{F}_{p_i}$ and $\mathbb{K} = \prod_{i=1}^m \mathbb{F}_{q_i}$ be two products of finite fields of pair-wise coprime order. Then the lattice of all $(\mathbb{F}, \mathbb{K})$-linearly closed clonoids is isomorphic to the direct product of the lattices of all $(\mathbb{F}_{p_i}, \mathbb{K})$-linearly closed clonoids with $1 \leq i \leq s$.

Upper bound for the cardinality of the lattice of all $(\mathbb{F},\mathbb{K})$-linearly closed clonoids

Theorem ’19 SF

Let $p_1,\dots,p_s,q_1,\dots,q_m$ be powers of distinct primes and let $\mathbb{F}=\prod_{i=1}^{s}\mathbb{F}_{p_i}$ and $\mathbb{K}=\prod_{j=1}^{m}\mathbb{F}_{q_j}$. Then the cardinality k of the lattice of all $(\mathbb{F},\mathbb{K})$-linearly closed clonoids $\mathcal{L}(\mathbb{F},\mathbb{K})$ is bounded by:

$$k \leq \prod_{i=1}^{s}\sum_{1\leq r\leq n}\binom{n}{r}_{p_i}, \tag{1}$$

where $n=\prod_{j=1}^{m} q_i$ and

$$\binom{n}{k}_q=\prod_{i=1}^{k}\frac{q^{n-k+i}-1}{q^i-1}. \tag{2}$$

Lattice of the $(\mathbb{F}_p, \mathbb{F}_q)$-linearly closed clonoids

Theorem '18 SF

Let p and q be powers of different primes. Then every $(\mathbb{F}_p, \mathbb{F}_q)$-linearly closed clonoid is generated by one unary function.

Lattice of the $(\mathbb{F}_p, \mathbb{F}_q)$-linearly closed clonoids

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Theorem '18 SF

Let p and q be powers of different primes. Let $\prod_{i=1}^{n} p_i^{k_i}$ be the factorization of the polynomial $g = x^{q-1} - 1$ in $\mathbb{F}_p[x]$ into its irreducible divisors. Then the number of distinct $(\mathbb{F}_p, \mathbb{F}_q)$-linearly closed clonoids is $2\prod_{i=1}^{n}(k_i + 1)$ and the lattice of all $(\mathbb{F}_p, \mathbb{F}_q)$-linearly closed clonoids, $\mathcal{L}(p, q)$, is isomorphic to

$$\mathbf{2} \times \prod_{i=1}^{n} \mathbf{C}_{k_i+1}.$$

Clones containing $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$

Main achievements:

We show that these clones are finitely many;

We provide an upper bound and a lower bound for the number of these clones;

We give sets of generators with a fixed bounded arity of the functions.

Known results

[1] E. Aichinger, P. Mayr, Polynomial clones on groups of order pq, in *Acta Mathematica Hungarica*, 2007.
(All 17 clones containing $\langle \mathbb{Z}_p \times \mathbb{Z}_q, +, (1,1) \rangle$);

[2] S. Kreinecker, Closed function sets on groups of prime order, in *Journal of Multiple-Valued Logic and Soft Computing*, 2019.
(All finitely many clones containing $\langle \mathbb{Z}_p, + \rangle$).

[3] P. Mayr, Polynomial clones on squarefree groups, in *Internat. J. Algebra Comput*, 2008.
(Proof that there are finitely many pol. expansions of a squarefree group).

Notation

We denote by $\mathcal{L}(\mathbb{Z}_s, +)$ the lattice of all clones containing the clone of $(\mathbb{Z}_s, +)$

We use boldface letters for vectors, e. g., $\boldsymbol{u} = (u_1, \dots, u_n)$ for some $n \in \mathbb{N}$

We write $[n] = \{1, \dots, n\}$ and $[n]_0 = [n] \cup \{0\}$

We consider $\prod_{i=1}^{m} \mathbb{Z}_{p_i}$ instead of $\mathbb{Z}_{p_1 \cdots p_m}$, since they are isomorphic

We write $\boldsymbol{x}^{\boldsymbol{m}}$ for $\prod_{i \in I} x_i^{m_i}$

We write $\mathbf{0}_k$ for the 0-vector of length k.

A general expression

Lemma

Let p and q be prime numbers. Then for every function f from $\mathbb{Z}_p^n \times \mathbb{Z}_q^n$ to $\mathbb{Z}_p \times \mathbb{Z}_q$ there exist two sequences of functions $\{f_{\boldsymbol{m}}\}_{\boldsymbol{m}\in[p-1]_0^n}$ from $\mathbb{Z}_q^n$ to $\mathbb{Z}_p$ and $\{g_{\boldsymbol{h}}\}_{\boldsymbol{h}\in[q-1]_0^n}$ from $\mathbb{Z}_p^n$ to $\mathbb{Z}_q$ such that f satisfies for all $\boldsymbol{x} \in \mathbb{Z}_p^n$, $\boldsymbol{y} \in \mathbb{Z}_q^n$:

$$f(\boldsymbol{x}, \boldsymbol{y}) = \Big(\sum_{\boldsymbol{m}\in[p-1]_0^n} f_{\boldsymbol{m}}(\boldsymbol{y})\boldsymbol{x}^{\boldsymbol{m}}, \sum_{\boldsymbol{h}\in[q-1]_0^n} g_{\boldsymbol{h}}(\boldsymbol{x})\boldsymbol{y}^{\boldsymbol{h}} \Big). \tag{3}$$

Induced functions

Let A be a set with a fixed element 0 and let $\mathbf{R}$ be a ring. For every polynomial $f \in \mathbf{R}^{A^n}[x_1, \ldots, x_k]$ of the form $f = \sum_{\boldsymbol{m} \in [u-1]_0^k} r_{\boldsymbol{m}} \boldsymbol{x}^{\boldsymbol{m}}$ we define its s-ary induced function $\overline{f}^{[s]}$: $R^s \times A^s \to R \times A$ by:

$$(\boldsymbol{x}, \boldsymbol{y}) \mapsto (\sum_{\boldsymbol{m} \in [u-1]_0^k} r_{\boldsymbol{m}}(\boldsymbol{z}) \prod_{i=1}^{k} x_i^{m_i}, 0),$$

with $s \geq k, n$ and $\boldsymbol{z} = (y_1, \ldots, y_n)$. We can observe that we induce also the functions $\{r_{\boldsymbol{m}}\}_{\boldsymbol{m} \in [u-1]_0^k}$ coefficients of monomials in f and for this reason we require $s \geq n$. From now on, when not specified, $s = \max(k, n)$

Properties of induced functions

Lemma

Let $d \in \mathbb{N}\backslash\{1\}$. Then for all $k, l \in \mathbb{N}$, for all $g \in \mathbb{Z}_p^{\prod_{i=1}^{m} \mathbb{Z}_{q_i}^l}$, and for all $\boldsymbol{m} \in [p-1]_0^k \backslash \{\mathbf{0}_k\}$ with $\sum_{i \in I} m_i = u$ congruent to d modulo $p-1$ it follows that:

$$\overline{r\boldsymbol{x}^{\boldsymbol{m}}} \in \mathrm{Clg}(\{\overline{rx_1 \cdots x_d}\}).$$

Properties of induced functions

Lemma

Let $p_1, \ldots, p_m$ distinct primes and let $n \in \mathbb{N}$, let $f\colon \prod_{i=1}^m \mathbb{Z}_{p_i}^n \to \prod_{i=1}^m \mathbb{Z}_{p_i}$ be an n-ary function. Let $h \in \mathbf{R}^A[X]$ be such that $\mathbf{R}^A = \mathbb{Z}_{p_1}^{\prod_{i=1}^m \mathbb{Z}_{q_i}^n}$ and $\overline{h} = g$. Let $h' = r\boldsymbol{x}^{\boldsymbol{m}}$ be a monomial of h with $\sum_{i \in I} m_i = d$. Then it follows that:

$$\overline{rx_1 \cdots x_d} \in \mathrm{Clg}(\{f\}).$$

Properties of induced functions

We can observe that composing $\overline{rx_1 \cdots x_d}$ with itself we obtain that

$$\overline{r^{l+1}x_1 \dots x_{d+l(d-1)}} \in \mathrm{Clg}(\overline{rx_1 \cdots x_d})$$

for all $l \in \mathbb{N}$. Since $r^p = r$ yields $r^{s(p-1)+1} = r$ for all $s \in \mathbb{N}$, it follows for $l = s(p-1)$ that

$$\overline{rx_1 \cdots x_{d+s(p-1)(d-1)}} \in \mathrm{Clg}(\overline{rx_1 \cdots x_d})$$

This yields

$$\overline{r\boldsymbol{x}^{\boldsymbol{m}}} \in \mathrm{Clg}(\overline{rx_1 \cdots x_d}).$$

An injective function

Theorem '19 SF

Let p and q be distinct prime numbers and let $\mathbf{Clo}^{\mathcal{L}}(\mathbb{Z}_{pq},+)$ be the lattice of all clones containing $\mathrm{Clo}(\mathbb{Z}_{pq},+)$. Then there is an injective function from $\mathbf{Clo}^{\mathcal{L}}(\mathbb{Z}_{pq},+)$ to the direct product of the lattice of all $(\mathbb{Z}_p,\mathbb{Z}_q)$-linearly closed clonoids, $\mathcal{L}(\mathbb{Z}_p,\mathbb{Z}_q)$, to the $p+1$ power and the lattice of all $(\mathbb{Z}_q,\mathbb{Z}_p)$-linearly closed clonoids, $\mathcal{L}(\mathbb{Z}_q,\mathbb{Z}_p)$, to the $q+1$ power, i. e:

$$\mathrm{Clo}^{\mathcal{L}}(\mathbb{Z}_{pq},+) \hookrightarrow \mathcal{L}(\mathbb{Z}_p,\mathbb{Z}_q)^{p+1} \times \mathcal{L}(\mathbb{Z}_q,\mathbb{Z}_p)^{q+1}.$$

An upper bound

Corollary '19 SF

Let p and q be distinct prime numbers. Let $\prod_{i=1}^{n} p_i^{k_i}$ and $\prod_{i=1}^{s} r_i^{d_i}$ be the factorizations of $g_p = x^{q-1} - 1$ in $\mathbb{Z}_p[x]$ and of $g_q = x^{p-1} - 1$ in $\mathbb{Z}_q[x]$ for irreducible p_i, q_i, respectively. Then:

$$2\left(\prod_{i=1}^{n}(k_i+1) + \prod_{i=1}^{s}(d_i+1)\right) - 1 \leq |\mathcal{L}(\mathbb{Z}_{pq}, +)| \leq$$

$$\leq 2^{p+q+2}\prod_{i=1}^{n}(k_i+1)^{p+1}\prod_{i=1}^{s}(d_i+1)^{q+1} \leq 2^{qp+q+p}.$$

Other parts of the lattice

Theorem '18 SF

Let p and q be distinct prime numbers. Then there is an injective function from the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$ that preserve π_1 and $[\pi_1, \pi_1] = 0$ to the direct product of the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_p, +)$ and the square of the lattice of all $(\mathbb{Z}_q, \mathbb{Z}_p)$-linearly closed clonoids.

Other parts of the lattice

Theorem '18 SF

Let p and q be distinct prime numbers. Then there is an injective function from the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$ that preserve π_1 and $[\pi_1, \pi_1] = 0$ to the direct product of the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_p, +)$ and the square of the lattice of all $(\mathbb{Z}_q, \mathbb{Z}_p)$-linearly closed clonoids.

Theorem '18 SF

Let p and q be two distinct prime numbers. Then the lattice of all $(\mathbb{Z}_p, \mathbb{Z}_q)$-linearly closed clonoids is embedded in the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$.

Clones of products of independent algebras

Theorem '18 SF

Let p and q be distinct prime numbers. Then there is an isomorphism between the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$ which preserve $\{\pi_1, \pi_2\}$ and the direct product of the lattices of all clones above $\mathrm{Clo}(\mathbb{Z}_p, +)$ and of all clones above $\mathrm{Clo}(\mathbb{Z}_q, +)$.

A set of generators

Corollary '19 SF

Let p and q be distinct prime numbers. Then the clones containing $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$ can be generated by a set of functions of arity at most $\max(p, q)$.

A set of generators

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Let p and q be distinct prime numbers. Then the clones containing $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$ can be generated by a set of functions of arity at most $\max(p, q)$.

Theorem

Let p and q be distinct prime numbers. Then a clone C containing $\mathrm{Clo}(\mathbb{Z}_{pq}, +)$ is generated by the sets of functions:

$$L := \bigcup_{i=0}^{p} \{\overline{rx_1 \cdots x_i} \mid r : \mathbb{Z}_q \to \mathbb{Z}_p, \overline{rx_1 \cdots x_i} \in C\}$$
$$R := \bigcup_{i=0}^{q} \{\overline{ry_1 \cdots y_i} \mid r : \mathbb{Z}_p \to \mathbb{Z}_q, \overline{ry_1 \cdots y_i} \in C\}.$$

Clones containing $\mathrm{Clo}(\mathbb{Z}_s, +)$

Main achievements if s is squarefree:

We show that these clones are finitely many;

We provide an upper bound and a lower bound for the number of these clones;

We give sets of generators with a fixed bounded arity of the functions;

We find a nice dichotomy for the clones of finite expansions of an abelian group.

Clones of expansions of a finite squarefree abelian group

Lemma

Let $p_1, \ldots, p_m$ be distinct prime numbers. Then for every function f from $\prod_{i=1}^m \mathbb{Z}_{p_i}^n$ to $\prod_{i=1}^m \mathbb{Z}_{p_i}$ there exist m sequences of functions $\{f_{\boldsymbol{h}_i}\}_{\boldsymbol{h}_i \in [p_i-1]_0^n}$ from $\prod_{j \in [m] \setminus \{i\}} \mathbb{Z}_{p_j}^n$ to $\mathbb{Z}_{p_i}$, for all $i \in [m]$, such that f satisfies for all $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \in \prod_{i=1}^m \mathbb{Z}_{p_i}^n$:

$$f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) = \left(\sum_{\boldsymbol{h}_1 \in [p_1-1]_0^n} f_{\boldsymbol{h}_1}(\boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)\boldsymbol{x}_1^{\boldsymbol{h}_1}, \ldots, \sum_{\boldsymbol{h}_m \in [p_m-1]_0^n} f_{\boldsymbol{h}_m}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m-1})\boldsymbol{x}_m^{\boldsymbol{h}_m}\right).$$

Embedding of the $(\mathbb{Z}_{p_i}, \mathbb{F}_i)$-linearly closed clonoids

Theorem '19 SF

Let $p_1, \dots, p_m$ be prime numbers and let $\mathbb{F}_1 = \prod_{i=2}^{m} \mathbb{Z}_{p_i}$. Then the lattice of all $(\mathbb{Z}_{p_1}, \mathbb{F}_1)$-linearly closed clonoids is embedded in the lattice of all clones above $\mathrm{Clo}(\mathbb{Z}_{p_1 \cdots p_m}, +)$.

Embedding of the $(\mathbb{Z}_{p_i}, \mathbb{F}_i)$-linearly closed clonoids

Let $\mathbb{F}_i = \prod_{j \in [m]\backslash\{i\}} \mathbb{Z}_{p_j}$. For all $i \in [m]$ and all $f \in \mathbb{Z}_{p_i}^{\mathbb{F}_i^n}$ we define $e_i(f) : \prod_{j=1}^m \mathbb{Z}_{p_j}^n \to \prod_{j=1}^m \mathbb{Z}_{p_j}$ by:

$$e_i(f)\colon (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \mapsto$$
$$(0_{\mathbb{Z}_{p_1}}, \dots, 0_{\mathbb{Z}_{p_{i-1}}}, f(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \dots, \boldsymbol{x}_m), 0_{\mathbb{Z}_{p_{i+1}}}, \dots, 0_{\mathbb{Z}_{p_m}})$$

for all $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \in \prod_{j=1}^m \mathbb{Z}_{p_j}^n$.

Embedding of the $(\mathbb{Z}_{p_1}, \mathbb{F}_1)$-linearly closed clonoids

We define γ_i from the lattice of all $(\mathbb{Z}_{p_i}, \mathbb{F}_i)$-linearly closed clonoids to the lattice of all clones containing $\mathrm{Clo}(\prod_{i\in[m]} \mathbb{Z}_{p_i}, +)$ such that for all $C \in \mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)$:

$$\gamma_i(C) := \bigcup_{n\in\mathbb{N}} \{e_i(g) + h_{(\boldsymbol{a}_1,\dots,\boldsymbol{a}_m)} \mid g \in C^{[n]}, (\boldsymbol{a}_1, \dots, \boldsymbol{a}_m) \in \prod_{j=1}^{m} \mathbb{Z}_{p_j}^n\}$$

where

$$h_{(\boldsymbol{a}_1,\dots,\boldsymbol{a}_m)} \colon (\boldsymbol{x}_1, \dots, \boldsymbol{x}_m) \mapsto (\langle \boldsymbol{a}_1, \boldsymbol{x}_1\rangle, \dots, \langle \boldsymbol{a}_m, \boldsymbol{x}_m\rangle)$$

Another embedding

Theorem '19 SF

Let $s = p_1 \cdots p_m$ be a product of distinct primes and let $\mathbb{F}_i = \prod_{j \in [m] \setminus \{i\}} \mathbb{Z}_{p_j}$ for all $1 \leq i \leq n$. Then there is an injective function from the lattice of all clones containing $\mathrm{Clo}(\mathbb{Z}_s, +)$, $\mathcal{L}(\mathbb{Z}_s, +)$, to the direct product of the lattices of all $(\mathbb{Z}_{p_i}, \mathbb{F}_i)$-linearly closed clonoids, $\mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)$, to the $p_i + 1$ power, i. e:

$$\mathcal{L}(\mathbb{Z}_s, +) \hookrightarrow \prod_{i=1}^{n} \mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)^{p_i+1}.$$

Another embedding

Let $s = p_1 \cdots p_m$ be a product of distinct prime numbers. Then for all $i \in [m]$ and $j \in [p_i]_0$ we define $\rho_{(i,j)} \colon \mathcal{L}(\mathbb{Z}_s, +) \to \mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)$ by:

$$\rho_{(i,j)}(C) := \bigcup_{n \in \mathbb{N}} \{f \colon \mathbb{F}_i^n \to \mathbb{Z}_{p_i} \mid \overline{f x_1 \cdots x_j} \in C\}$$

for all $C \in \mathcal{L}(\mathbb{Z}_s, +)$

Let $\rho : \mathcal{L}(\mathbb{Z}_s, +) \to \prod_{i=1}^{m} \mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)^{p_i+1}$ be defined by $\rho(C) = (\rho_{(1,0)}(C), \ldots, \rho_{(1,p_1)}(C), \ldots, \rho_{(m,0)}(C), \ldots, \rho_{(m,p_m)}(C))$.

Bounds for the cardinality

Corollary '19 SF

Let $s = p_1 \cdots p_m$ be a product of distinct primes and let $\mathbb{F}_i = \prod_{j \in [n] \setminus \{i\}} \mathbb{Z}_{p_j}$. Then the number of clones containing $\mathrm{Clo}(\mathbb{Z}_s, +)$ is bounded by:

$$\sum_{i=1}^{m} |\mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)| - m + 1 \leq |\mathcal{L}(\mathbb{Z}_s, +)| \leq \prod_{i=1}^{m} |\mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)|^{p_i+1}.$$

where $\mathcal{L}(\mathbb{Z}_{p_i}, \mathbb{F}_i)$ is the lattice of all $(\mathbb{Z}_{p_i}, \mathbb{F}_i)$-linearly closed clonoids.

Bounds for the cardinality

Corollary '19 SF

Let $s = p_1 \cdots p_m \in \mathbb{N}$ be a product of distinct primes and let $\mathbb{F}_i = \prod_{j \in [m] \setminus \{i\}} \mathbb{Z}_{p_j}$ for all $i \in [m]$. Then the number of clones containing $\mathrm{Clo}(\mathbb{Z}_s, +)$ is bounded by:

$$|\mathcal{L}(\mathbb{Z}_s, +)| \leq \prod_{i=1}^{m} \Big(\sum_{1 \leq r \leq n_i} \binom{n_i}{r}_{p_i} \Big)^{p_i+1}$$

where $n_i = \prod_{j \in [m] \setminus \{i\}} p_j$ and

$$\binom{n}{k}_q = \prod_{i=1}^{k} \frac{q^{n-k+i} - 1}{q^i - 1}.$$

A set of generators

Theorem '19 SF

Let $s = p_1 \cdots p_m$ be a product of distinct prime numbers. Then the clones containing $\mathrm{Clo}(\mathbb{Z}_s, +)$ can be generated by a set of functions of arity at most $\max(p_1, \ldots, p_m)$.

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$$S_i := \bigcup_{j=0}^{p_i} \{\overline{rx_1 \cdots x_j} \mid r\colon \mathbb{F}_i \to \mathbb{Z}_{p_i}, \overline{rx_1 \cdots x_j} \in C\}.$$

A dichotomy

Theorem '19 SF

Let $\mathbf{G}$ be a finite abelian group. Then $\mathbf{G}$ has finitely many expansions up to term equivalence or, equivalently, the lattice of all clones containing $\mathrm{Clo}(G, +, -, 0)$ is finite if and only if $\mathbf{G}$ is of squarefree order.

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THANK YOU!!!!