Characterizing $[\alpha, \beta] = 0$ using Kiss terms

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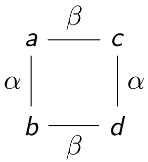
Rectangles

Suppose $\alpha, \beta \in \operatorname{Con} \mathbf{A}$.

An (α, β)-*rectangle* is a 4-tuple $(a, b, c, d) \in A^4$ with



$R(\alpha, \beta) := \{\text{all } (\alpha, \beta)\text{-rectangles}\} \leq \mathbf{A}^4$.

Observe: $$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta) \implies (c, c') \in \alpha \cap \beta.$$

(TC) commutator

Again $\alpha, \beta \in \operatorname{Con} \mathbf{A}$.

$$\operatorname{Const}(\alpha,\beta) := \left\{ \begin{bmatrix} x & x \\ y & y \end{bmatrix} : x \overset{\alpha}{\equiv} y \right\} \cup \left\{ \begin{bmatrix} u & v \\ u & v \end{bmatrix} : u \overset{\beta}{\equiv} v \right\} \subseteq R(\alpha,\beta)$$

$$M(\alpha,\beta) := \text{the subalgebra of } \mathbf{A}^4 \text{ generated by } \operatorname{Const}(\alpha,\beta) \leq \underline{R}(\alpha,\beta)$$

$$= \left\{ \begin{bmatrix} t(\mathbf{x},\mathbf{u}) & t(\mathbf{x},\mathbf{v}) \\ t(\mathbf{y},\mathbf{u}) & t(\mathbf{y},\mathbf{v}) \end{bmatrix} : t \text{ a term, } x_i \overset{\alpha}{\equiv} y_i,\ u_j \overset{\beta}{\equiv} v_j \right\}.$$

(the "α, β-matrices")

"$[\alpha,\beta] = 0$" means

$$\boxed{a = c \iff b = d} \quad \text{for all } \begin{bmatrix} a \overset{\theta}{\equiv} c \\ b \overset{\theta}{\equiv} d \end{bmatrix} \in M(\alpha,\beta).$$

$[\alpha,\beta] =$ least θ ,

Difference term varieties

Definition

A term $p(x, y, z)$ is a *difference term* for a variety $\mathcal{V}$ if it satisfies

$$p(x, x, y) \approx y \quad \text{throughout } \mathcal{V} \tag{1}$$

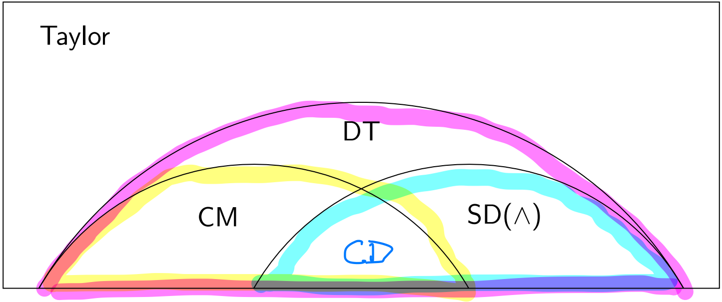
$$p(x, y, y) \overset{[\theta,\theta]}{\equiv} x \quad \text{whenever } (x, y) \in \theta \in \operatorname{Con} \mathbf{A} \text{ in } \mathcal{V} \tag{2}$$

$\mathcal{V}$ is a *difference term* (DT) *variety* if it has a difference term.

Examples of DT varieties

0. Any variety with a Maltsev term.
1. Any CM variety (Herrmann, Gumm).
2. Any SD($\wedge$) variety: they satisfy $[\alpha, \beta] = \alpha \cap \beta$, so $p(x, y, z) := z$ is a difference term.

Every DT variety is a Taylor variety.



Mantra

If a statement is true for all CM varieties **and** all SD(∧) varieties, then it is probably true for all DT varieties.

Definition

A term $q(x, y, z, w)$ is a *Kiss term* for a variety $\mathcal{V}$ if it satisfies

$$q(x, y, x, y) \approx x \quad \text{throughout } \mathcal{V} \tag{3}$$

$$q(x, x, y, y) \approx y \quad \text{throughout } \mathcal{V} \tag{4}$$

$$q(a, b, c, d) \overset{[\alpha,\beta]}{\equiv} q(a, b, c', d) \quad \text{whenever } \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta)$$

$$\text{where } \alpha, \beta \in \text{Con } \mathbf{A} \text{ and } \mathbf{A} \in \mathcal{V} \tag{5}$$

Examples of varieties having a Kiss term

1. Any CM variety (Kiss).
2. Any SD($\wedge$) variety: $q(x, y, z, w) := z$ is a Kiss term.
3. Any DT variety (Lipparini).

Theorem (Kiss)

Let $\mathcal{V}$ be a CM variety with Kiss term $q(x, y, z, w)$. Suppose $\mathbf{A} \in \mathcal{V}$ and $\alpha, \beta \in \operatorname{Con} \mathbf{A}$. Then $[\alpha, \beta] = 0$ if and only if

$\alpha \wedge \beta$

(K1) *q restricted to $R(\alpha, \beta)$ does not depend on its 3rd variable:*

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in R(\alpha, \beta) \implies q(a, b, c, d) = q(a, b, c', d).$$

(K2) *q restricted to $R(\alpha, \beta)$ is a homomorphism $\mathbf{R}(\alpha, \beta) \to \mathbf{A}$.*

Kiss's Theorem is also true for SD($\wedge$) varieties with $q = z$.

RTP: $\alpha \wedge \beta = 0 \iff$ (K1).

($\Rightarrow$)

($\Leftarrow$) Assume (K1). $(a, b) \in \alpha \wedge \beta$.

$\begin{bmatrix} a & a \\ a & a \end{bmatrix}, \begin{bmatrix} a & b \\ a & a \end{bmatrix} \in R(\alpha, \beta) \overset{(K1)}{\implies} a = b$. So $\alpha \wedge \beta = 0$.

So Kiss's Theorem should also be true for DT varieties.

"Lemma 6.2" (KSW 2016)

Every DT variety $\mathcal{V}$ has a Kiss term q such that in any $\mathbf{A} \in \mathcal{V}$,

$[\alpha, \beta] = 0 \iff$

(K1) q restricted to $R(\alpha, \beta)$ does not depend on z, and

(K2) $q : \mathbf{R}(\alpha, \beta) \to \mathbf{A}$ is a homomorphism.

This lemma was a key step in our proof of Park's Conjecture for DT varieties.

The story gets interesting

Aug 2021: the authors of ALVIN enlist volunteers to help them proofread the forthcoming volumes 2 and 3.

- Our extension of Kiss's Theorem is in Vol. 3.

Oct 2021: Peter Mayr notices that the ALVIN proof of our theorem is bogus. Alerts Ralph Freese.

- Ralph and Peter study our published proof of the theorem. They find that our proof is also bogus!

Back to the beginning

$[\alpha,\beta]=0 \iff$ (K1), (K2).

Fix a DT variety $\mathcal{V}$ and a Kiss term q.

To extend Kiss's characterization of $[\alpha,\beta]=0$ to $\mathcal{V}$, the nontrivial implication is:

$$[\alpha,\beta]=0 \implies \text{(K2)}:\ q|_{R(\alpha,\beta)} \text{ is a homomorphism} \qquad (*)$$

Reduction. $\mathcal{V} \models (*)$ if (and only if) in $\mathcal{V}$ we have

$[\alpha,\beta]=0 \implies \exists\, \Delta$ satisfying $M(\alpha,\beta) \subseteq \Delta \leq \mathbf{R}(\alpha,\beta)$ and

(R1) For all $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$, $a=c \iff b=d$, (centrality)

(R2) For all $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$ there exists c' with $\begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta$. (ampleness)

$\exists\ \Delta$ satisfying $M(\alpha,\beta) \subseteq \Delta \leq \mathbf{R}(\alpha,\beta)$ and

(R1) $\forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta, \quad a = c \iff b = d,$ (centrality)

(R2) $\forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$ there ~~exists~~ $\exists!$ c' with $\begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta$. (ampleness)

Lemma. (In a DT variety), if such Δ exists then it is unique.

Proof. Fix Kiss term q. Fix Δ, α, β, $[\alpha,\beta]=0$. Assume Δ exists.

Claim 1. $q(abcd) = c \quad \forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$.

Pf. $\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} b & b \\ b & b \end{bmatrix}, \begin{bmatrix} c & c \\ d & d \end{bmatrix}, \begin{bmatrix} d & b \\ d & b \end{bmatrix} \in \Delta$. Apply q.

(given $\in \Delta$; Const (α,β))

$$\Delta \ni \begin{bmatrix} q(abcd) & q(cbcb) \\ q(bbdd) & q(dbdb) \end{bmatrix} = \begin{bmatrix} q(abcd) & c \\ d & d \end{bmatrix} \overset{(R1)}{\Longrightarrow} q(abcd) = c.$$

Claim 2. $\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta \implies c = c'$.

Pf. Kiss term $\implies$ $\underset{c}{q(abcd)} \overset{[\alpha,\beta]=0}{\equiv} \underset{c'}{q(abc'd)}$.

$\Delta \leq \mathbf{R}(\alpha,\beta)$, (Clm.1) $q(a,b,c,d) = c$ for all $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$, and

(R2!) $\forall \begin{bmatrix} a & * \\ b & d \end{bmatrix} \in R(\alpha,\beta)$ $\exists!\ c$ with $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta$.

Claim 3. $\Delta = \left\{ \begin{bmatrix} a & q(abcd) \\ b & d \end{bmatrix} : \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta) \right\}$.

Pf. ✓ ($\supseteq$) Let $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha,\beta)$.

RTP: $\begin{bmatrix} a & q(abcd) \\ b & d \end{bmatrix} \in \Delta$.

(R2!) let c' be unique --- $\begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta$. $\subseteq R(\alpha,\beta)$

Kiss term $\Rightarrow$ $q(abcd) \overset{[\alpha,\beta]=0}{\equiv} q(abc'd) = c'$. So $\begin{bmatrix} a & q(abcd) \\ b & d \end{bmatrix} \in \Delta$.

($\subseteq$). $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta \subseteq R(\alpha,\beta)$

RHS $\ni \begin{bmatrix} a & q(abcd) \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Questions

Q. Why not just define Δ as in last Claim and prove it works?

(Can't prove it is a subalgebra, without knowing $q|_{R(\alpha,\beta)}$ is a hom.)

Q. What Δ works in CM varieties?

$\Delta_{\alpha,\beta}$:= the "horizontal transitive closure" of $M(\alpha, \beta)$ works (Kiss).

Q. What Δ works in SD($\wedge$) varieties?

$$\Delta := R(\alpha, \beta) \text{ works.}$$

Q. What formulaic subalgebra specializes to $\Delta_{\alpha,\beta}$ in CM varieties and to $R(\alpha, \beta)$ in SD($\wedge$) varieties?

Answer

Given $\alpha, \beta \in \text{Con}\,\mathbf{A}$,

$\Delta^*_{\alpha,\beta}$:= "horizontal and vertical transitive closure" of $M(\alpha, \beta)$.

Channeling:

- Moorhead (2021): "2-dimensional congruence"
- Janelidze & Pedicchio (2001): "double congruence"

Theorem (KSW, 202?)

For all DT varieties, $[\alpha, \beta] = 0 \implies \Delta^*_{\alpha,\beta}$ works!

In particular:

1. In CM varieties, $\Delta^*_{\alpha,\beta} = \Delta_{\alpha,\beta}$
2. In SD($\wedge$) varieties, $\Delta^*_{\alpha,\beta} = R(\alpha, \beta)$.

Theorem (expanded)

In DT varieties, $[\alpha, \beta] = 0 \implies M(\alpha, \beta) \subseteq \Delta^*_{\alpha,\beta} \leq \mathbf{R}(\alpha, \beta)$ and

(R1) $\forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \Delta^*_{\alpha,\beta}, \quad a = c \iff b = d,$ (centrality)

(R2) $\forall \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in R(\alpha, \beta) \ \exists\, c'$ with $\begin{bmatrix} a & c' \\ b & d \end{bmatrix} \in \Delta^*_{\alpha,\beta}.$ (ampleness)

Proof hints.

(R1): routine proof using difference term; or use $[\alpha, \beta] = [\alpha, \beta]_\ell$.

(R2): long calculation using the Maltsev condition for DT varieties.

"Extremely complicated, magical"

Putting everything together, we have proved:

Corollary (KSW, new)

Kiss's characterization of $[\alpha, \beta] = 0$ extends to DT varieties (really!).

Final comment

Kiss's proof that $\Delta_{\alpha,\beta}$ satisfies (R2) in CM varieties was high-level, using the modular law in $\mathrm{Con}\,\boldsymbol{\alpha}$ and properties of the commutator deducible from a difference term.

Our proof that $\Delta^*_{\alpha,\beta}$ satisfies (R2) in DT varieties is syntactic, using the Maltsev condition for DT varieties.

Question

Are there properties of congruences (or double congruences) in DT varieties that could lead to a nicer proof?

References

G. Janelidze and M.C. Pedicchio, *Pseudogroupoids and commutators*, TAC **8** (2001), 408–456.

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A. Moorhead, *Supernilpotent Taylor algebras are nilpotent*, TAMS **374** (2021), 1229–1276.

Thank you!