Inherently nonfinitely based nonassociative algebras

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Work in progress. . .

Fix a field $\mathbb{F}$.

"(Associative) $\mathbb{F}$-algebra":

$\approx$ ring also having the structure of an $\mathbb{F}$-vector space.

Examples:

- $\mathbb{F}[x]$, $\mathbb{F}[x_1, \ldots, x_n]$
- $\mathbb{F}\langle x_1, \ldots, x_n \rangle$
- $\mathbf{End}(V)$ for any $\mathbb{F}$-vector space V

Given an $\mathbb{F}$-algebra $\mathbf{R}$, a subset $U \subseteq R$, and a term $t = t(x_1, \ldots, x_n)$, I will say "t **vanishes on** U" and write

$$\mathbf{R} \models t \overset{U}{\approx} 0$$

to mean $t^{\mathbf{R}}(u_1, \ldots, u_n) = 0$ for all $u_1, \ldots, u_n \in U$.

"Nonassociative $\mathbb{F}$-algebra":

$\approx$ non-unital $\mathbb{F}$-algebra minus associativity of multiplication.

$=$ $\mathbb{F}$-vector space with a <u>bilinear</u> binary operation (multiplication), i.e.,

$$\begin{aligned}(x+\lambda y)z &\approx xz+\lambda(yz)\\ x(y+\lambda z) &\approx xy+\lambda(xz).\end{aligned}$$

Examples abound:

- Choose a vector space V.
- Choose a basis: $e_1, e_2, \ldots$
- Choose values for products $e_i e_j$.
- Extend the product operation bilinearly.

In this talk we only consider nonassoc. $\mathbb{F}$-algebras satisfying

$$(xy)z \approx 0 \qquad \text{“2-step right nilpotence”}$$

In such algebras, nonzero nested products must be right-associated:

$$a_1(a_2(a_3(\cdots a_{n-1}(a_n b))\cdots))).$$

Main example

(L'vov): Given a vector space V, **L'vov's algebra** is

$$(\mathrm{End}(V) \oplus V, \cdot)$$

with multiplication (for $f, g \in \mathrm{End}(V)$ and $x, y \in V$)

$$(f + x)(g + y) = f(y).$$

We have right nilpotency:

$$\Big((f + x)(g + y)\Big)(h + z) = \Big(0 + f(y)\Big)(h + z) = 0.$$

Related examples

More generally: Given any subspace $A \leq (\mathrm{End}(V), +)$, we can form

$$(A \oplus V, \cdot) \quad \leq \quad (\mathrm{End}(V) \oplus V, \cdot).$$

Even more generally: given any $\phi : A \xrightarrow{\text{lin}} \mathrm{End}(V)$, we can define $(A \oplus V, \cdot_\phi)$ in the obvious way.

Theorem (L'vov, 1978, building on Polin, 1976)

If $\mathbb{F}$ *is finite and* $\dim(V) = 2$*, then* $(\mathrm{End}(V) \oplus V, \cdot)$ *is not finitely based.*

Theorem (Isaev, 1989)

If $\mathbb{F}$ *is finite,* $V = \mathbb{F}^2$*, and* UT *is the vector space of* 2×2 *upper-triangular matrices over* $\mathbb{F}$*, then* $(UT \oplus \mathbb{F}^2, \cdot)$ *is* **inherently** *nonfinitely based.*

Problems:

- Which other $(A \oplus V, \cdot)$ are inherently nonfinitely based (INFB)?
- Does INFB imply INFB_{fin} for these algebras?

(INFB_{fin} = "INFB in the finite sense")

2-sorted variants

$\mathrm{Lvov}(V) := (\mathrm{End}(V) \oplus V, \cdot)$

$\mathrm{Lvov}_2(V) :=$ a **2-sorted** algebra

- Sorts: $\mathrm{End}(V)$ and V.
- $\mathbb{F}$-vector space operations on each sort.
- Binary operation $\mathrm{End}(V) \times V \to V$.

$\mathrm{Lvov}_2^-(V)$:

- Like $\mathrm{Lvov}_2(V)$, but
- Throw away the vector space operations on the first sort $\mathrm{End}(V)$, **except** the constant 0.

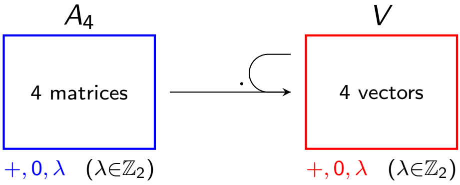
Example

Let $\mathbb{F} = \mathbb{Z}_2$, $V = (\mathbb{Z}_2)^2$, and

$$A_4 = \left\{ \begin{bmatrix} a & a \\ 0 & b \end{bmatrix} : \; a, b \in \mathbb{Z}_2 \right\}.$$

1. $(A_4 \oplus V, \cdot)$ is a 4-dimensional subalgebra of Isaev's $(UT \oplus (\mathbb{Z}_2)^2, \cdot)$.

2. $(A_4, V, \cdot)^+$ has 2 sorts and 9 operations:



3. $(A_4, V, \cdot)^-$ is like $(A_4, V, \cdot)^+$ but with 3 fewer operations on sort 1.

Lemma

Suppose $\mathbb{F}$ *is finite,* V *is finite dimensional, and* $A \leq \mathrm{End}(V)$*.*

For each of the properties FB, INFB, *and* INFB_{fin}, *either* *all* *of*

$$(A \oplus V, \cdot),\ (A, V, \cdot)^{+},\ \textit{and}\ (A, V, \cdot)^{-}$$

have the property, or *none* *have it.*

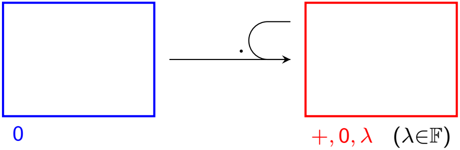
Ingredients in proof:

- Every nonassociative $\mathbb{F}$-algebra satisfying $(xy)z \approx 0$ embeds "nicely" into some $(A \oplus V, \cdot_\phi)$;
- Every model $(X, V, \cdot)^{-}$ of some minimal axioms is a subreduct of $(\mathrm{Free}(X), V, \cdot_\phi)^{+}$;

and syntactical considerations.

The variety $\Omega_{\mathbb{F}}^{-}$

Fix finite $\mathbb{F}$. Call the sorts of $(A, V, \cdot)^{-}$ "sort 1" (blue) and "sort 2" (red).

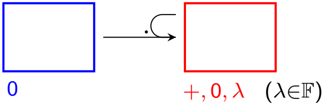


Let $\Omega_{\mathbb{F}}^{-}$ be the variety axiomatized by the following (finite) set of axioms:

- $\mathbb{F}$-vector space axioms for V using $+, 0, \lambda$ $(\lambda \in \mathbb{F})$.
- Linearity of $\cdot$ in 2nd variable:

$$\left.\begin{aligned} x(y+z) &\approx xy + xz \\ x(\lambda y) &\approx \lambda(xy) \qquad (\lambda \in \mathbb{F}). \end{aligned}\right. \qquad \Delta_{\mathbb{F}}^{-}$$

- Action of 0: $\quad 0y \approx 0.$



Modulo $\Delta_{\mathbb{F}}^{-}$, the terms in this signature are:

Sort 1: $x_1, x_2, \ldots$ and 0.

Sort 2: $\lambda_1 y_1 + \cdots + \lambda_k y_k$

xy

$x_1(x_2(x_2 y))$

$x_2(x_1(x_2(\lambda y + \mu z))) \quad \longrightarrow \quad \lambda(x_2(x_1(x_2 y))) + \mu(x_2(x_1(x_2 z)))$

Fix a variable y of sort 2. Consider terms like

$$\lambda(x_1(x_2(x_1(x_3 y)))) + \mu(x_2(x_2(x_1 y))) + x_1(x_3 y) + y$$

Our notation for this term:

$$[\,\lambda x_1 x_2 x_1 x_3 + \mu x_2 x_2 x_1 + x_1 x_3 + 1\,]y.$$

$$[\,\lambda x_1x_2x_1x_3 + \mu x_2x_2x_1 + x_1x_3 + 1\,]y.$$

Inside the brackets:

- an $\mathbb{F}$-linear combination of *words in* $\{x_1, x_2, x_3\}^*$
- I.e., an element of $\mathbb{F}\langle x_1, x_2, x_3\rangle$.

For any set X of (sort 1) variables:

1. To every $p \in \mathbb{F}\langle X\rangle$ we can assign a sort 2 term $[p]y$ in X and y.
2. Conversely, every sort 2 term in X and y is equivalent mod $\Delta_{\mathbb{F}}^{-}$ to some $[p]y$ ($p \in \mathbb{F}\langle X\rangle$).

Fact 1

Every subvariety of $\Omega_{\mathbb{F}}^{-}$ is axiomatized by $\Delta_{\mathbb{F}}^{-}$ and a subset of:

- $x \approx 0$
- $y \approx 0$
- Identities of the form $[p]y \approx 0$ where $p \in \mathbb{F}\langle X\rangle$.

Evaluating $[p]y$ in $(A, V, \cdot)^-$

Fix $p = p(x_1, \ldots, x_n) \in \mathbb{F}\langle X \rangle$.

Let $A \leq \text{End}(V)$.

Given $a_1, \ldots, a_n \in A$ and $v \in V$,

$$([p]y)^{(A,V,\cdot)^-}(a_1, \ldots, a_n; v) \;=\; \underbrace{p(a_1, \ldots, a_n)}_{\text{eval. in } \mathbf{End}(V)}(v)$$

Hence:

Fact 2

Let $A \leq \text{End}(V)$ and $p \in \mathbb{F}\langle X \rangle$.

$$(A, V, \cdot)^- \models [p]y \approx 0 \quad \iff \quad p(a_1, \ldots, a_n) = 0 \text{ for all } a_i \in A$$
$$\iff \quad \mathbf{End}(V) \models p \overset{A}{\approx} 0.$$

$$(A, V, \cdot)^- \models [p]y \approx 0 \iff \mathbf{End}(V) \models p \overset{A}{\approx} 0.$$

In particular: when $A = \mathrm{End}(V)$,

$$(\mathrm{End}(V), V, \cdot)^- \models [p]y \approx 0 \iff \mathbf{End}(V) \models p \approx 0.$$

Conundrum

When $V = \mathbb{F}^2$, $\mathbf{End}(V)$ has a finite basis

$$\{p_1 \approx 0,\ p_2 \approx 0\} \cup \{\mathbb{F}\text{-algebra axioms}\}.$$

Question: Why isn't

$$\{[p_1]y \approx 0,\ [p_2]y \approx 0\} \cup \Delta_{\mathbb{F}}^-$$

a finite basis for $(\mathrm{End}(V), V, \cdot)^-$?

Answer: Derivations work differently.

- Only variable-to-(variable $\cup \{0\}$) substitutions in the p's are allowed.

Let X be countably infinite.

1. The $(\omega, 1)$-generated free algebra in $\Omega_{\mathbb{F}}^{-}$ is

$$X \cup \{0\} \hookrightarrow \mathbb{F}\langle X \rangle$$

2. The subvarieties of $\Omega_{\mathbb{F}}^{-}$ are in 1-1 correspondence* with the ideals of $\mathbb{F}\langle X \rangle$ which are closed under variable-to-(variable $\cup \{0\}$) substitutions.

3. Given $n \geq 1$ and $\Sigma \subseteq \mathbb{F}\langle X \rangle$, the $(n, 1)$-generated free algebra in the variety defined by $\Delta_{\mathbb{F}}^{-} \cup \{[p]y \approx 0 \,:\, p \in \Sigma\}$ is

$$\begin{matrix} x_1, \ldots, x_n \\ 0 \end{matrix} \hookrightarrow \mathbb{F}\langle x_1, \ldots, x_n \rangle / \mathcal{I}_v^n(\Sigma)$$

where $\mathcal{I}_v^n(\Sigma)$ is the smallest ideal of $\mathbb{F}\langle x_1, \ldots, x_n \rangle$ containing all variable-to-$\{x_1, \ldots, x_n, 0\}$ substitution instances of members of Σ.

* Plus the variety defined by $\Delta_{\mathbb{F}}^{-} \cup \{x \approx 0\}$, and the trivial variety.

Strategy to determine whether $(A \oplus V, \cdot)$ is FB or INFB

1. Determine which $p \in \mathbb{F}\langle X \rangle$ vanish on A.
   - $\Delta_{\mathbb{F}}^- \cup \{[p]y \approx 0 \,:\, p \text{ vanishes on } A\}$ is a basis for $(A, V, \cdot)^-$.

2. Find an "efficient" set Σ of p vanishing on A which still gives a basis.

3. (If Σ is infinite): for each finite subset $\Sigma_0 \subseteq \Sigma$, somehow prove that for large enough n,
   - $\mathcal{I}_v^n(\Sigma_0) \neq \mathcal{I}_v^n(\Sigma)$ (proving NFB), or
   - $\mathcal{I}_v^n(\Sigma_0)$ has infinite index in $\mathbb{F}\langle x_1, \ldots, x_n \rangle$ (proving INFB).

4. (If successful in step 3): Crack open a beverage of your choice and celebrate!

5. (If greedy): Distill the proof in (3) to get a tidy, direct HSP argument of INFB not involving Σ, $\mathbb{F}\langle X \rangle$, 2-sorted algebras, etc.

What we can do

1. For Isaev's algebra $(UT \oplus (\mathbb{Z}_2)^2, \cdot)$, we can do everything.
    - And almost everything with $\mathbb{Z}_2$ replaced by any finite $\mathbb{F}$.
    - To be fair, Isaev essentially did steps 1–3.

2. Recall the subalgebra $(A_4 \oplus (\mathbb{Z}_2)^2, \cdot)$ where

$$A_4 = \left\{ \begin{bmatrix} a & a \\ 0 & b \end{bmatrix} : \; a, b \in \mathbb{Z}_2 \right\}.$$

   We've done steps 1–2; have a nice set Σ. Can't prove NFB yet.

3. The nightmare $(A_5 \oplus (\mathbb{Z}_2)^2, \cdot)$ where

$$A_5 = \left\{ \begin{bmatrix} a & a \\ b & 0 \end{bmatrix} : \; a, b \in \mathbb{Z}_2 \right\}.$$

Some problems

1. Is $(A_4 \oplus (\mathbb{Z}_2)^2, \cdot)$ INFB?
2. (If yes) Why?
3. Is every $(A \oplus V, \cdot)$ either FB or INFB?
4. Suppose $\mathcal{I}$ is an ideal of $\mathbb{F}\langle x_1, \ldots, x_n \rangle$ closed under variable-to-(variable $\cup \{0\}$) substitutions. If $\mathcal{I}$ has infinite index, must there exist left ideals containing $\mathcal{I}$ having arbitrarily large finite index?