Varieties generated by finite simple algebras

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Vague question

What can we say about the structure of (sub)direct products?

$\mathbf{C} \leq \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is a **subdirect product** (denoted $\mathbf{C} \leq_{\mathrm{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$) if all projections $\pi_i \colon \mathbf{C} \to \mathbf{A}_i$ are onto.

$\mathbf{A}$ is **subdirectly irreducible (SI)** if it has a unique minimal congruence (the **monolith**).

Subdirect Representation Theorem (Birkhoff)

Every algebra is a subdirect product of its subdirectly irreducible quotients.

No surprises here

A finite subdirect product of simple groups is isomorphic to a direct product. More generally:

Theorem (Foster-Pixley)

Let $\mathbf{C} \leq_{\mathrm{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ for simple $\mathbf{A}_1, \ldots, \mathbf{A}_n$ in a congruence permutable (CP) variety. Then

$$\mathbf{C} \cong \mathbf{A}_{i_1} \times \cdots \times \mathbf{A}_{i_m} \text{ for some } 1 \leq i_1 < \cdots < i_m \leq n.$$

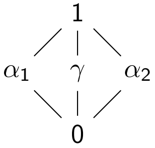
Weird things happening, 1

Direct products of simple algebras can have unexpected quotients:

Example (abelian factors, CP)

Let $\mathbf{A} = (\mathbb{Z}_2, +, 0, 1)$. Then $\mathbf{A}^2$ has a skew congruence γ with blocks $\{00, 11\}$, $\{01, 10\}$ and

$$\mathbf{A}^2/\gamma \cong (\mathbb{Z}_2, +, 0, 0) \ncong \mathbf{A}.$$



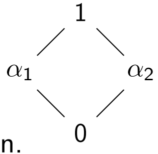
Weird things happening, 2

Subdirect products of simple algebras may not be direct:

Example (congruence modular (CM), not CP)

Let $\mathbf{L} = (\{0,1\}, \wedge, \vee)$ be the 2-element lattice.

$$C = \{00, 01, 11\} \leq_{\mathrm{sd}} \mathbf{L}^2$$

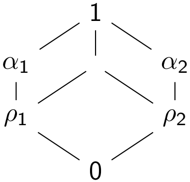


is not a direct power of $\mathbf{L}$. Every quotient of $\mathbf{C}$ is a projection.

Weird things happening, 3

Example (not CM)

Let $\mathbf{S} = (\mathbb{Z}_2, \cdot)$ the 2-element semilattice. $\mathrm{Con}(\mathbf{S}^2) =$



ρ_1 is induced by the ideal $\{00, 01\}$ in $\mathbf{S}^2$

ρ_2 is induced by the ideal $\{00, 10\}$

Question

Can we describe congruences of (sub)direct products in a restricted setting, e.g., in congruence modular (CM) varieties.

Advantages:

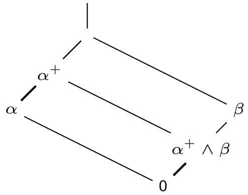
- modular law (no pentagons in congruence lattices)
- powerful commutator theory for congruences generalized from group theory

Congruences of subdirect products in CM varieties

Splitting the congruence lattice

Lemma 1

Let $\mathbf{C} \leq_{\mathrm{sd}} \mathbf{A} \times \mathbf{B}$ with projection kernels α, β in a CM variety.
Assume $\mathbf{A}$ is SI with nonabelian monolith, and α^+ is the unique minimal congruence above α.
Then $\gamma \leq \alpha$ or $\gamma \geq \alpha^+ \wedge \beta$ for every $\gamma \in \mathrm{Con}(\mathbf{C})$.



Proof.

- If $\alpha^+ \wedge \beta \leq \alpha$, then $\alpha^+ \wedge \beta = \alpha \wedge \beta = 0 \leq \gamma$.
- So assume $\alpha^+ \wedge \beta, \gamma \not\leq \alpha$. Then $(\alpha^+ \wedge \beta) \vee \alpha = \alpha^+ \leq \gamma \vee \alpha$.
- Using monotonicity and join distributivity of the commutator

$$[\alpha^+, \alpha^+] \leq [\gamma \vee \alpha, (\alpha^+ \wedge \beta) \vee \alpha] = \underbrace{[\gamma, \alpha^+ \wedge \beta]}_{\leq \gamma \wedge (\alpha^+ \wedge \beta)} \vee \underbrace{[., \alpha] \vee [., \alpha] \vee [\alpha, \alpha]}_{\leq \alpha}$$

- Since $[\alpha^+, \alpha^+] \not\leq \alpha$, this yields $0 \neq \gamma \wedge (\alpha^+ \wedge \beta) = \alpha^+ \wedge \beta$

More factors, more problems

Lemma 2

Let $\mathbf{C} \leq_{\text{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \times \mathbf{B}$ with projection kernels $\alpha_1, \ldots, \alpha_n, \beta$ in a CM variety. Assume $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are SI with nonabelian monoliths.
If $\gamma \not\leq \alpha_i$ for any $i \in [n]$, then $\gamma \geq \bigwedge_{i\in[n]} \alpha_i^+ \wedge \beta$.

Proof sketch (induction on n).

Base $n = 1$: Lemma 1.
Step $n = 1 \to 2$: By induction assumption, $\gamma \not\leq \alpha_i$ for $i = 1, 2$ yields

$$\gamma \geq (\alpha_1^+ \wedge \alpha_2 \wedge \beta) \vee (\alpha_1 \wedge \alpha_2^+ \wedge \beta).$$

By modularity

$$\begin{aligned}(\alpha_1^+ \wedge \alpha_2 \wedge \beta) \vee [\alpha_1 \wedge \alpha_2^+ \wedge \beta] &= [(\alpha_1^+ \wedge \alpha_2 \wedge \beta) \vee \alpha_1] \wedge \alpha_2^+ \wedge \beta \\ &= \alpha_1^+ \wedge \underbrace{[(\alpha_2 \wedge \beta) \vee \alpha_1]}_{\geq \alpha_1^+,\ \text{else}\ \alpha_2 \wedge \beta = 0} \wedge \alpha_2^+ \wedge \beta\end{aligned}$$

Note

Let $\mathbf{C} \leq_{\mathrm{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n \times \mathbf{B}$ in a CM variety with $\mathbf{A}_i$ SI with nonabelian monoliths as in Lemma 2.
Then the interval $I(0, \bigwedge_{i \in [n]} \alpha_i^+ \wedge \beta)$ is a Boolean sublattice of $\mathrm{Con}(\mathbf{C})$.

Corollary

Let $\mathbf{C} \leq_{\mathrm{sd}} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ with simple nonabelian $\mathbf{A}_i$ in a CM variety.
Then $\mathrm{Con}(\mathbf{C})$ is a Boolean lattice and every quotient of $\mathbf{C}$ is a projection on some factors.

Proof.

By Lemma 2 every $\gamma \in \mathrm{Con}(\mathbf{C})$ is an intersection of projection kernels. $\square$

Beyond CM varieties

Question

Is there a finite (sub)direct product of simple nonabelian algebras with a nontrivial abelian quotient?

Finitely generated varieties

Some notation

A **variety** is a class of algebras of fixed type defined by equations (equivalently closed under homomorphic images $\mathbb{H}$, subalgebras $\mathbb{S}$ and direct products $\mathbb{P}$).

For a class of algebras K of the same type, $V(K) := \mathbb{HSP}(K)$ is the **variety generated by** K.

For a variety V and $k \in \mathbb{N}$, let $\mathbf{F}_k(V)$ denote the **free algebra** over $x_1, \ldots, x_k$ in V.

Subdirect representation of free algebras

Lemma

Let $\mathbf{A}$ be finite, $V := V(\mathbf{A})$ and $W := V(\mathbf{B} \, : \, \mathbf{B} < \mathbf{A})$. Then for every $k \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that

$$\mathbf{F}_k(V) \leq_{\mathrm{sd}} \mathbf{A}^d \times \mathbf{F}_k(W).$$

Proof.

$V := V(\mathbf{A})$ and $W := V(\mathbf{B} : \mathbf{B} < \mathbf{A})$.

- Let $\mathbf{F} := \mathbf{F}_k(V)$ free over $x_1, \ldots, x_k$. For $a = (a_1, \ldots, a_k)$ in A^k, define

$$\psi_a \colon \mathbf{F} \to \mathbf{A} \text{ by } x_i \mapsto a_i \text{ for } i \leq k,$$

$$\rho := \bigwedge\{\ker \psi_a \; : \; \langle a \rangle = A\}, \qquad \sigma := \bigwedge\{\ker \psi_a \; : \; \langle a \rangle < A\}.$$

- If $(s(x_1, \ldots, x_k), t(x_1, \ldots, x_k)) \in \rho \wedge \sigma$, then

$$s(a_1, \ldots, a_k) = t(a_1, \ldots, a_k) \text{ for all } (a_1, \ldots, a_k) \in A^k.$$

  Hence $s = t$ in $\mathbf{F}$ and $\rho \wedge \sigma = 0$.

- Thus

$$\mathbf{F} \leq_{\mathrm{sd}} \underbrace{\mathbf{F}/\rho}_{\leq_{\mathrm{sd}} A^d} \times \underbrace{\mathbf{F}/\sigma}_{\cong \mathbf{F}_k(W)}$$

$\square$

Can $\mathbf{A}$ be in the variety generated by its proper subalgebras?

Sure, e.g., $\mathbf{A} = (\mathbb{Z}_6, +)$.

Can this happen for simple $\mathbf{A}$?

Lemma (Kearnes, Szendrei)

If a finite simple $\mathbf{A}$ is in $V(\mathbf{B} : \mathbf{B} < \mathbf{A})$, then $\mathbf{A}$ has TCT-type 5.

Proof.

- If $\mathbf{A}$ has type 1 or 2 (i.e. $\mathbf{A}$ is abelian), all its subalgebras are trivial (Valeriote, 1990).
- If $\mathbf{A}$ has type 3 or 4, then $\mathbf{A} \in \mathbb{HSP}(\mathbf{B} : \mathbf{B} < \mathbf{A})$ yields $\mathbf{A} \in \mathbb{HS}(\mathbf{B})$ for some $\mathbf{B} < \mathbf{A}$ (Hobby, McKenzie, 1988).

$\square$

A bad example

Example

Murskii's groupoid M $= (\{0, 1, 2\}, \cdot)$ is defined by

$$\begin{array}{c|ccc} \cdot & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 2 & 2 \end{array}$$

$\mathbf{M}^2$ has a congruence γ with a single nontrivial class $\{00, 01, 02, 10, 20\}$.
$\mathbf{A} := \mathbf{M}^2/\gamma$ is simple and $\mathbf{M} \to \mathbf{A},\ x \mapsto (x, x)/\gamma$ is an embedding.
Hence $\mathbf{A}$ is in the variety generated by its subalgebra $\mathbf{M}$.

Question

Is there a finite simple $\mathbf{A}$ in the variety W generated by its proper subalgebras where W omits type 1?

Main result

Theorem

Let $\mathbf{A}$ be a finite SI with nonabelian monolith μ in a CM variety. Then $V := V(\mathbf{A})$ has a unique maximal subvariety U (generated by $\mathbf{A}/\mu$ and all proper subalgebras of $\mathbf{A}$).

Proof.

1. $U < V$ because $\mathbf{A}$ is not in any CM variety generated by finitely many strictly smaller algebras by

- the generalized Jónsson Lemma for CM varieties or
- elementary commutator calculations (omitted).

Theorem

Let $\mathbf{A}$ be a finite SI with nonabelian monolith μ in a CM variety.
Then $V := V(\mathbf{A})$ has a unique maximal subvariety U (generated by $\mathbf{A}/\mu$ and all proper subalgebras of $\mathbf{A}$).

Proof.

2. Every proper subvariety of V is below U:

- Let $\mathbf{C} \in V$ be finite such that $V(\mathbf{C}) \neq V$. Then $\mathbf{C}$ is a quotient of

$$\mathbf{F}_k(V) \leq_{\mathrm{sd}} \mathbf{A}^d \times \mathbf{F}_k(W)$$

  for some $k \in \mathbb{N}$ and $W = V(\mathbf{B} : \mathbf{B} < \mathbf{A})$.
- Since $\mathbf{A}$ is not a quotient of $\mathbf{C}$, Lemma 2 yields that $\mathbf{C}$ is a quotient of

$$\mathbf{F}_k(V)/\mu^d \times 0_{\mathbf{F}_k(W)} \leq_{\mathrm{sd}} (\mathbf{A}/\mu)^d \times \mathbf{F}_k(W).$$

- Hence every finite $\mathbf{C}$ from a proper subvariety of V is in U.
  Since V is locally finite, every proper subvariety of V is in U.

Questions on the lattice of subvarieties of $V(\mathbf{A})$

Problem 10.1 in Sixty-four Problems in Universal Algebra (2001)

Is every subvariety of a finitely generated CM variety finitely generated?

Yes, for varieties with cube term (Aichinger, M 2016).

Problem

Is there a finitely generated variety with cube term (or CM variety) with an infinite antichain of subvarieties?