Varieties generated by finite simple algebras

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Vague question

What can we say about the structure of (sub)direct products?

\[ C \leq A_1 \times \cdots \times A_n \text{ is a subdirect product} \quad \text{(denoted } C \leq_{sd} A_1 \times \cdots \times A_n \text{)} \]

if all projections \( \pi_i : C \to A_i \) are onto.

\textbf{A is subdirectly irreducible (SI) if it has a unique minimal congruence (the monolith).}

\textbf{Subdirect Representation Theorem (Birkhoff)}

Every algebra is a subdirect product of its subdirectly irreducible quotients.
A finite subdirect product of simple groups is isomorphic to a direct product. More generally:

**Theorem (Foster-Pixley)**

Let $C \leq_{sd} A_1 \times \cdots \times A_n$ for simple $A_1, \ldots, A_n$ in a congruence permutable (CP) variety. Then

$$C \cong A_{i_1} \times \cdots \times A_{i_m} \text{ for some } 1 \leq i_1 < \cdots < i_m \leq n.$$
Direct products of simple algebras can have unexpected quotients:

**Example (abelian factors, CP)**

Let $A = (\mathbb{Z}_2, +, 0, 1)$. Then $A^2$ has a skew congruence $\gamma$ with blocks $\{00, 11\}$, $\{01, 10\}$ and

$$A^2/\gamma \cong (\mathbb{Z}_2, +, 0, 0) \not\cong A.$$
Weird things happening, 2

Subdirect products of simple algebras may not be direct:

Example (congruence modular (CM), not CP)

Let $L = (\{0, 1\}, \wedge, \vee)$ be the 2-element lattice. 

$$C = \{00, 01, 11\} \leq_{sd} L^2$$

is not a direct power of $L$. Every quotient of $C$ is a projection.
Example (not CM)

Let $S = (\mathbb{Z}_2, \cdot)$ the 2-element semilattice. $\text{Con}(S^2) =$

$\rho_1$ is induced by the ideal $\{00, 01\}$ in $S^2$

$\rho_2$ is induced by the ideal $\{00, 10\}$
Question

Can we describe congruences of (sub)direct products in a restricted setting, e.g., in congruence modular (CM) varieties.

Advantages:

- modular law (no pentagons in congruence lattices)
- powerful commutator theory for congruences generalized from group theory
Congruences of subdirect products in CM varieties
Lemma 1

Let $C \leq_{sd} A \times B$ with projection kernels $\alpha, \beta$ in a CM variety.
Assume $A$ is SI with nonabelian monolith, and $\alpha^+$ is the unique minimal congruence above $\alpha$.
Then $\gamma \leq \alpha$ or $\gamma \geq \alpha^+ \land \beta$ for every $\gamma \in \text{Con}(C)$.

Proof.

- If $\alpha^+ \land \beta \leq \alpha$, then $\alpha^+ \land \beta = \alpha \land \beta = 0 \leq \gamma$.
- So assume $\alpha^+ \land \beta, \gamma \not\leq \alpha$. Then $(\alpha^+ \land \beta) \lor \alpha = \alpha^+ \leq \gamma \lor \alpha$.
- Using monotonicity and join distributivity of the commutator

$$[\alpha^+, \alpha^+] \leq [\gamma \lor \alpha, (\alpha^+ \land \beta) \lor \alpha] = [\gamma, \alpha^+ \land \beta] \lor [., \alpha] \lor [., \alpha] \lor [\alpha, \alpha] \leq \gamma \land (\alpha^+ \land \beta) \leq \alpha$$

- Since $[\alpha^+, \alpha^+] \not\leq \alpha$, this yields $0 \neq \gamma \land (\alpha^+ \land \beta) = \alpha^+ \land \beta$.
More factors, more problems

Lemma 2
Let \( C \subseteq_{sd} A_1 \times \cdots \times A_n \times B \) with projection kernels \( \alpha_1, \ldots, \alpha_n, \beta \) in a CM variety. Assume \( A_1, \ldots, A_n \) are SI with nonabelian monoliths. If \( \gamma \not\subseteq \alpha_i \) for any \( i \in [n] \), then \( \gamma \geq \bigwedge_{i \in [n]} \alpha_i^+ \land \beta \).

Proof sketch (induction on \( n \)).
Base \( n = 1 \): Lemma 1.
Step \( n = 1 \rightarrow 2 \): By induction assumption, \( \gamma \not\subseteq \alpha_i \) for \( i = 1, 2 \) yields
\[
\gamma \geq (\alpha_1^+ \land \alpha_2 \land \beta) \lor (\alpha_1 \land \alpha_2^+ \land \beta).
\]
By modularity
\[
(\alpha_1^+ \land \alpha_2 \land \beta) \lor [\alpha_1 \land \alpha_2^+ \land \beta] = [(\alpha_1^+ \land \alpha_2 \land \beta) \lor \alpha_1] \land \alpha_2^+ \land \beta
\]
\[
= \alpha_1^+ \land [(\alpha_2 \land \beta) \lor \alpha_1] \land \alpha_2^+ \land \beta
\]
\[
\geq \alpha_1^+ \text{, else } \alpha_2 \land \beta = 0
\]
Note
Let $C \leq_{sd} A_1 \times \cdots \times A_n \times B$ in a CM variety with $A_i$ SI with nonabelian monoliths as in Lemma 2. Then the interval $I(0, \bigwedge_{i \in [n]} \alpha_i^+ \land \beta)$ is a Boolean sublattice of $\text{Con}(C)$.

Corollary
Let $C \leq_{sd} A_1 \times \cdots \times A_n$ with simple nonabelian $A_i$ in a CM variety. Then $\text{Con}(C)$ is a Boolean lattice and every quotient of $C$ is a projection on some factors.

Proof.
By Lemma 2 every $\gamma \in \text{Con}(C)$ is an intersection of projection kernels. □
Beyond CM varieties

Question

Is there a finite (sub)direct product of simple nonabelian algebras with a nontrivial abelian quotient?
Finitely generated varieties
A **variety** is a class of algebras of fixed type defined by equations (equivalently closed under homomorphic images $\mathbb{H}$, subalgebras $\mathbb{S}$ and direct products $\mathbb{P}$).

For a class of algebras $K$ of the same type, $V(K) := \text{HSP}(K)$ is the **variety generated by** $K$.

For a variety $V$ and $k \in \mathbb{N}$, let $F_k(V)$ denote the **free algebra** over $x_1, \ldots, x_k$ in $V$. 

Lemma

Let $A$ be finite, $V := V(A)$ and $W := V(B : B < A)$. Then for every $k \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that

$$F_k(V) \leq_{sd} A^d \times F_k(W).$$
Proof.

\( V := V(A) \) and \( W := V(B : B < A) \).

- Let \( F := F_k(V) \) free over \( x_1, \ldots, x_k \). For \( a = (a_1, \ldots, a_k) \) in \( A^k \), define \( \psi_a : F \rightarrow A \) by \( x_i \mapsto a_i \) for \( i \leq k \),

\[ \rho := \bigwedge \{ \ker \psi_a : \langle a \rangle = A \}, \quad \sigma := \bigwedge \{ \ker \psi_a : \langle a \rangle < A \}. \]

- If \( (s(x_1, \ldots, x_k), t(x_1, \ldots, x_k)) \in \rho \wedge \sigma \), then \( s(a_1, \ldots, a_k) = t(a_1, \ldots, a_k) \) for all \( (a_1, \ldots, a_k) \in A^k \).

Hence \( s = t \) in \( F \) and \( \rho \wedge \sigma = 0 \).

- Thus

\[
F \leq_{sd} \frac{F}{\rho} \times \frac{F}{\sigma} \leq_{sd} A^d \cong F_k(W)
\]
Can $A$ be in the variety generated by its proper subalgebras?

Sure, e.g., $A = (\mathbb{Z}_6, +)$.

Can this happen for simple $A$?

**Lemma (Kearnes, Szendrei)**

If a finite simple $A$ is in $V(B : B < A)$, then $A$ has TCT-type 5.

**Proof.**

- If $A$ has type 1 or 2 (i.e. $A$ is abelian), all its subalgebras are trivial (Valeriote, 1990).
- If $A$ has type 3 or 4, then $A \in HSP(B : B < A)$ yields $A \in HS(B)$ for some $B < A$ (Hobby, McKenzie, 1988).
A bad example

Example

Murskii’s groupoid \( \mathbf{M} = (\{0, 1, 2\}, \cdot) \) is defined by

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
2 & 0 & 2 & 2 \\
\end{array}
\]

\( \mathbf{M}^2 \) has a congruence \( \gamma \) with a single nontrivial class \( \{00, 01, 02, 10, 20\} \). \( \mathbf{A} := \mathbf{M}^2 / \gamma \) is simple and \( \mathbf{M} \to \mathbf{A}, \ x \mapsto (x, x) / \gamma \) is an embedding. Hence \( \mathbf{A} \) is in the variety generated by its subalgebra \( \mathbf{M} \).

Question

Is there a finite simple \( \mathbf{A} \) in the variety \( \mathcal{W} \) generated by its proper subalgebras where \( \mathcal{W} \) omits type 1?
Main result

Theorem

Let $A$ be a finite SI with nonabelian monolith $\mu$ in a CM variety. Then $V := V(A)$ has a unique maximal subvariety $U$ (generated by $A/\mu$ and all proper subalgebras of $A$).

Proof.

1. $U < V$ because $A$ is not in any CM variety generated by finitely many strictly smaller algebras by
   - the generalized Jónsson Lemma for CM varieties or
   - elementary commutator calculations (omitted).
Theorem

Let $A$ be a finite SI with nonabelian monolith $\mu$ in a CM variety. Then $V := V(A)$ has a unique maximal subvariety $U$ (generated by $A/\mu$ and all proper subalgebras of $A$).

Proof.

2. Every proper subvariety of $V$ is below $U$:

- Let $C \in V$ be finite such that $V(C) \neq V$. Then $C$ is a quotient of

$$F_k(V) \leq_{sd} A^d \times F_k(W)$$

for some $k \in \mathbb{N}$ and $W = V(B : B < A)$.

- Since $A$ is not a quotient of $C$, Lemma 2 yields that $C$ is a quotient of

$$F_k(V)/\mu^d \times 0_{F_k(W)} \leq_{sd} (A/\mu)^d \times F_k(W).$$

- Hence every finite $C$ from a proper subvariety of $V$ is in $U$.

Since $V$ is locally finite, every proper subvariety of $V$ is in $U$. 
Questions on the lattice of subvarieties of $V(A)$

Problem 10.1 in Sixty-four Problems in Universal Algebra (2001)

Is every subvariety of a finitely generated CM variety finitely generated?

Yes, for varieties with cube term (Aichinger, M 2016).

Problem

Is there a finitely generated variety with cube term (or CM variety) with an infinite antichain of subvarieties?