Relative lengths of Maltsev conditions

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Outline

Relative lengths of Maltsev conditions (PALS 2022)

1. Introduction. An old problem by Gumm and Lakser, Taylor, Tschantz and why I believe that such problems are interesting
2. An old problem by A. Day and a partial solution: the many ways in which congruence distributivity implies congruence modularity
3. Gumm, near-unanimity and directed terms. Further problems

Thanks to the organizers (and the speakers) for the interesting series of seminars!

Thanks also for the invitation!

(Full details about the relevant definitions shall be given shortly!)

- Recall that (Day 1969) a variety $\mathcal{V}$ is congruence modular if and only if $\mathcal{V}$ has n Day terms, for some $n \in \mathbb{N}$.
- Recall that (Gumm 1981) a variety $\mathcal{V}$ is congruence modular if and only if $\mathcal{V}$ has k Gumm terms, for some $k \in \mathbb{N}$.

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Henceforth, for every n, there is some k such that every variety with n Day terms has k Gumm terms.

A similar remark applies to all the similar situations we shall describe below.

Congruence modular varieties are characterized both by Day and Gumm terms. Moreover, there is a theoretical connection between the possible numbers of such terms.
In practice,

- THEOREM (Lakser, Taylor, Tschantz 1985) Every variety with $n+1$ Day terms has $k = n^2 - n + 2$ Gumm terms.

Can we do better?

- PROBLEM (implicit in Gumm 1983; explicitly, LTT 1985) For every n, evaluate the best possible value of k as above.

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(Minor improvements are possible, but I do not know of any significant improvement.)

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  - Some further "miraculous" characterizations of congruence modular varieties (not only Gumm's): Nation's two variable characterization (1974, see also Day Freese 1980), Freese and Jónsson's equivalence with the Arguesian identity (1976), Tschantz (1985), Dent, Kearnes, Szendrei (2012) etc.

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  - Surprising results for varieties satisfying a non trivial idempotent Maltsev condition (Hobby, McKenzie 1988, Kearnes, Kiss 2013).
  - The existence of the weakest nontrivial idempotent Maltsev condition (Siggers 2010, Olšák 2017).
  - An equivalent characterization of congruence distributivity by means of "directed terms" (Kazda, Kozik, McKenzie, Moore 2018, details below).
  - Etc.

- While lots of astonishing things are known about the interplay of distinct Maltsev conditions. . .
- . . . really little is known about the relative lengths of such conditions. For example,
  - as we mentioned, little is known about LTT's problem of evaluating the smallest possible number of Gumm terms from Day terms (only recent results about the converse).

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  - Day 1969 also showed that if some variety $\mathcal{V}$ has $n+1$ Jónsson terms, then $\mathcal{V}$ has $2n$ Day terms. He asked whether this is best possible. As far as I know, a solution has never been proposed before.

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    - Day 1969 also showed that if some variety $\mathcal{V}$ has $n+1$ Jónsson terms, then $\mathcal{V}$ has $2n$ Day terms. He asked whether this is best possible. As far as I know, a solution has never been proposed before.
    - Which is the relationship between the number of Jónsson and directed Jónsson terms? (Kazda, Kozik, McKenzie, Moore 2018).
    - Etc.

Recall LTT's Problem: for every n, evaluate the best possible value k such that every variety with n Day terms has k Gumm terms.

- A solution to this and similar problems is supposed to provide
  - either interesting exotic examples of congruence modular and distributive varieties, or
  - more refined structure theorems.

For example (Lipparini 2020), if some *congruence distributive* variety $\mathcal{V}$ has k Gumm terms, then $\mathcal{V}$ has $k+1$ Jónsson terms (Jónsson terms characterize congruence distributive varieties; more details below).

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If the bound found by LTT can be improved, the above influence will show to be tighter. If the bound found by LTT cannot be improved, the influence remains quite loose.

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back to the beginnings!

Recall that a *variety* is a class of algebraic structures (algebras, for short) closed under products, substructures and homomorphic images. Equivalently, by Birkhoff's theorem, a variety is a class of algebras which is equationally definable.

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A *congruence* on some algebra is an equivalence relation which is compatible, that is, the kernel of some homomorphism.

Two congruences α and β *permute* if $\alpha \circ \beta = \beta \circ \alpha$. A variety $\mathcal{V}$ is *congruence permutable* if all congruences of every algebra in $\mathcal{V}$ pairwise permute.

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A variety is *congruence distributive (modular)* if the congruence lattices of all its algebras are distributive (modular).

- THEOREM (Maltsev 1954) A variety $\mathcal{V}$ is congruence permutable if and only if there is a term in the language of $\mathcal{V}$ such that the equations

$$x = t(x, y, y), \quad t(x, x, y) = y$$

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hold throughout $\mathcal{V}$.

- This applies to groups, quasigroups, rings, Boolean algebras... The property is preserved by expansions, hence the theorem applies also to every algebra with additional structure.

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- More importantly, as far as the problems here are concerned, Maltsev Theorem initiated a flourishing study of similar conditions (with a difference!)

- THEOREM (Jónsson 1967) A variety $\mathcal{V}$ is congruence distributive if and only if there exist some $n \in \mathbb{N}$ and ternary terms $t_0, \dots, t_n$ in the language of $\mathcal{V}$ such that

$$
\begin{aligned}
x &= t_0(x, y, z), & \\
x &= t_h(x, y, x), & \text{for } 0 \leq h \leq n, \\
t_h(x, x, z) &= t_{h+1}(x, x, z), & \text{for } h \text{ even, } 0 \leq h < n \\
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  are equations valid in $\mathcal{V}$.

Notice: in contrast with Maltsev theorem, here n varies!

Conditions of the above form are called *Maltsev conditions*.
If the number of terms (and their arities) are fixed, as in Maltsev theorem, one speaks of *strong* Maltsev conditions. Here we always deal with Maltsev (not strong) conditions.

- REMARK.
- The sequence $t_0, \dots, t_n$ consists of $n+1$ terms.
- In Jónsson condition the terms t_0 and t_n are trivial projections, hence there are $n-1$ nontrivial terms.
- A variety satisfying Jónsson condition for some specific n is usually said to be n-distributive.
- Henceforth sometimes the counting conventions clash!

- THEOREM 1 (Day 1969) A variety $\mathcal{V}$ is congruence modular if and only if there exist some $m \in \mathbb{N}$ and 4-ary terms $u_0, \dots, u_m$ such that

$$
\begin{aligned}
x &= u_0(x,y,z,w), & & \\
x &= u_k(x,y,y,x), & & \text{for } 0 \le k \le m, \\
u_k(x,x,w,w) &= u_{k+1}(x,x,w,w), & & \text{for even } k,\ 0 \le k < m, \\
u_k(x,y,y,w) &= u_{k+1}(x,y,y,w), & & \text{for odd } k,\ 0 \le k < m, \\
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(Despite the appearances, there are some similarities with Jónsson conditions. Think of the second and the third variables here as some kind of a "doubling" of the Jónsson second variable.)

- THEOREM 2 (Day 1969) If $\mathcal{V}$ is a congruence distributive variety, as witnessed by Jónsson terms $t_0, \ldots, t_n$, then $\mathcal{V}$ is congruence modular (obvious! but also) witnessed by Day terms $u_0, \ldots, u_{2n-1}$.

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The proof is easy (at least in hindsight).

- *Proof.* Given Jónsson terms $t_0, \ldots, t_n$, define

$$\begin{aligned} u_0(x,y,z,w) &= x \\ u_1(x,y,z,w) &= t_1(x,y,w), \\ u_2(x,y,z,w) &= t_1(x,z,w), \\ u_3(x,y,z,w) &= t_2(x,z,w), \\ u_4(x,y,z,w) &= t_2(x,y,w), \\ u_5(x,y,z,w) &= t_3(x,y,w), \\ u_6(x,y,z,w) &= t_3(x,z,w), \\ &\ldots \end{aligned}$$

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u_3(x,y,z,w) &= t_2(x,z,w), \\
u_4(x,y,z,w) &= t_2(x,y,w), \\
u_5(x,y,z,w) &= t_3(x,y,w), \\
u_6(x,y,z,w) &= t_3(x,z,w), \\
&\dots
\end{aligned}$$

PROBLEM (Day 1969) Is the result best possible?

Summarizing: from Jónsson terms $t_0, \dots, t_n$ we can get Day terms $u_0, \dots, u_{2n-1}$. Can we do better?

- Yes, if n is odd!

Proof. $\dots$

$$\begin{aligned}
u_{2n-6}(x,y,z,w) &= t_{n-3}(x,y,w),\\
u_{2n-5}(x,y,z,w) &= t_{n-2}(x,y,w),\\
u_{2n-4}(x,y,z,w) &= t_{n-2}(x,z,w),\\
u_{2n-3}(x,y,z,w) &= t_{n-1}(\mathbf{y},\mathbf{z},w),\\
u_{2n-2}(x,y,z,w) &= w.
\end{aligned}$$

Indeed, since n is odd, $u_{2n-4}(x,x,w,w) = t_{n-2}(x,w,w) = t_{n-1}(x,w,w) = u_{2n-3}(x,x,w,w)$
and $u_{2n-3}(x,y,y,w) = t_{n-1}(y,y,w) = t_n(y,y,w) = w$.

The idea appears in Lakser, Taylor, Tschantz 1985 in a different context.

Apparently, the connection with Day's problem is not mentioned in LTT.

REMARK. We do not even need to assume $x = t_{n-1}(x, y, x)$!

- PROBLEM. We do not know whether in the case n odd Day's result can be further improved.

However…

Summarizing: from Jónsson terms $t_0, \dots, t_n$ we can get Day terms $u_0, \dots, u_{2n-1}$. For short, every n-distributive variety is $2n-1$-modular.

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Sketch of proof. As already noticed by Day, lattices are 2-distributive (this is the same as to say that lattices have a majority term t_1; recall that t_0 and t_2 in Jónsson conditions are projections).
On the other hand, it is easy to see that if a variety $\mathcal{V}$ is 2-modular, then $\mathcal{V}$ is congruence permutable. Lattices are not congruence permutable, hence lattices are 3-modular, by Day's result, but not 2-modular (here $n = 2$, thus $2n - 1 = 3$).

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This is the case $n = 2$. The proof now proceeds by induction.

The induction step. I. Relabeling the operations.

Suppose that $n \geq 4$, n is even, and we have constructed some variety $\mathcal{V}_{n-2}$ which is $n-2$-distributive (hence $2n-5$-modular, by Day's result) but not $2n-6$-modular.

We want to construct some variety $\mathcal{V}_n$ which is n-distributive but not $2n-2$-modular.

The induction step. I. Relabeling the operations.

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We want to construct some variety $\mathcal{V}_n$ which is n-distributive but not $2n-2$-modular.

It is no loss of generality to assume that the Jónsson terms of $\mathcal{V}_{n-2}$ are operations; actually, it is no loss of generality to assume that $\mathcal{V}_{n-2}$ has only the Jónsson operations.

The induction step. I. Relabeling the operations (continued)

We have assumed that $\mathcal{V}_{n-2}$ has only the Jónsson operations for $n-2$-distributivity, say, $s_0, \dots, s_{n-2}$.

Relabel the operations as $t_1 = s_0, \dots, t_{n-1} = s_{n-2}$ and take t_0 to be the projection onto the first coordinate, t_n to be the projection onto the third coordinate. We get a variety, call it $\mathcal{V}^+_{n-2}$ with operations $t_0, \dots, t_n$.

So far, so good!

The induction step. I. Relabeling the operations (continued)

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So far, so good!

Of course, $\mathcal{V}_{n-2}$ and $\mathcal{V}^+_{n-2}$ are mutually interpretable, hence $\mathcal{V}^+_{n-2}$ alone does not suffice for our purposes.

The induction step. II. Constructing another variety.
Let $n \geq 4$. Consider the variety generated by term-reducts of Boolean algebras with ternary operations

$$\begin{aligned} t_0(x,y,z) &= x, \\ t_1(x,y,z) &= x(y'+z), \\ t_2(x,y,z) &= xz, \\ &\cdots \\ t_{n-2}(x,y,z) &= xz, \\ t_{n-1}(x,y,z) &= z(y'+x), \\ t_n(x,y,z) &= z, \end{aligned}$$

where $+$ and $\cdot$ are the lattice operations and $'$ is complement. Let $\mathcal{B}_n$ denote this variety.

The induction step. III. Joining the varieties.

Let $\mathcal{V}_n$ be the join of $\mathcal{V}^+_{n-2}$ and $\mathcal{B}_n$.

Using some elaborate construction, it can be shown that $\mathcal{V}_n$ is not $2n-4$-modular.

The induction step. III. Joining the varieties.

Let $\mathcal{V}_n$ be the join of $\mathcal{V}^+_{n-2}$ and $\mathcal{B}_n$.

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Here we run into troubles!

First, we wanted to construct some n-distributive variety which is not $2n-2$-modular, but we only got "not $2n-4$-modular" (the bound is shifted by 2, but we need a shift by 4).

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Even worse!
$\mathcal{V}_n$ is **not** necessarily n-distributive.

Indeed, we have shifted the Jónsson operations by 1, e. g., $t_2 = s_1$. Hence even and odd are exchanged and we get a condition different from Jónsson's (of course, if n is kept fixed).

Interlude: the ALVIN condition.
Let us recall Jónsson condition.

- THEOREM (Jónsson 1967) A variety $\mathcal{V}$ is congruence distributive if and only if there exist some $n \in \mathbb{N}$ and ternary terms $t_0, \dots, t_n$ such that

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\begin{aligned}
x &= t_0(x, y, z), && \\
x &= t_h(x, y, x), && \text{for } 0 \le h \le n, \\
t_h(x, x, z) &= t_{h+1}(x, x, z), && \text{for } h \textbf{ even}, 0 \le h < n \\
t_h(x, z, z) &= t_{h+1}(x, z, z), && \text{for } h \textbf{ odd}, 0 \le h < n \\
t_n(x, y, z) &= z. &&
\end{aligned}
$$

If odd and even are exchanged, we get a different condition!

The ALVIN condition

- The Jónsson condition with odd and even exchanged (McKenzie, McNulty, Taylor 1987) has been called the ALVIN condition.
- For n odd we have that n-ALVIN and n-distributive are equivalent.
- This is not the case when n is even (Freese, Valeriote 2009).

Restructuring the proof.

Let us return to our sketch of proof.

We wanted "not $2n-2$-modular", but we only got "not $2n-4$-modular".
We wanted an n-distributive variety $\mathcal{V}_n$, but we have got an n-ALVIN variety.

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So far, our proof is failing badly. How can we recover?

We need to broaden our perspective!
It is not enough to study the exact relationships between n-distributive and m-modular: we need to deal simultaneously with n-distributive, m-modular, n-ALVIN and m-reversed-modular. The last condition means the Day's condition in which odd and even are exchanged.

Restructuring the proof (continued).

- So we actually need a double induction.
- We need to construct simultaneously, for each even $n \geq 2$,
  - an n-distributive variety which is not $2n-1$-reversed-modular (in particular, as we wanted, not $2n-2$-modular), and also
  - an n-alvin variety which is not $2n-3$-modular.

The base cases are the variety of lattices and the variety of Boolean algebras.

(By the way, the restructured argument proves much more!)

Restructuring the proof. The revised induction step.

For the induction step, we take unions of appropriate varieties, as above.
From an $n-2$-ALVIN not $2n-7$-modular variety we construct an n-distributive variety which is not $2n-1$-reversed-modular.

In the other case, from an $n-2$-distributive variety which is not $2n-5$-reversed-modular we construct an n-ALVIN not $2n-3$-modular variety.

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The shift in the modularity level is 6 in the former case and 2 in the latter case. On average, we get a shift by 8 each time n is increased by 4, in agreement with what we wanted to prove.

Full details in Lipparini 2019.

Further results. The above arguments apply to many more situations.

- (Gumm, LTT) Suppose that $n \geq 4$, n even. Every variety with Gumm terms $t_0, \ldots, t_n$ has Day terms $u_0, \ldots, u_{2n-2}$.

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- Further generalizations, even dealing with conditions which do not imply congruence modularity.
    - In particular, we can frequently deal with "defective" conditions in which some equations are not assumed (in conditions like Jónsson's or Day's).
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  - Such "specular" conditions might have independent interest, for example, they appear in Chiccho's Thesis (2018) in connection with m-permutability.

We now give the explicit definition of Gumm and directed terms.

THEOREM (Gumm 1981) A variety $\mathcal{V}$ is congruence modular if and only if, for some n, there are ternary terms $t_0, \ldots, t_n$ such that

$$\begin{aligned} x &= t_0(x, y, z), & \\ x &= t_h(x, y, x), & \text{for } 1 < h \leq n, \\ t_h(x, x, z) &= t_{h+1}(x, x, z), & \text{for } h \text{ odd, } 0 \leq h < n \\ t_h(x, z, z) &= t_{h+1}(x, z, z), & \text{for } h \text{ even, } 0 \leq h < n \\ t_n(x, y, z) &= z. & \end{aligned}$$

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This is a slightly different formulation, in comparison with Gumm original definition. Possibly, this formulation first appeared in print in LTT. It allows a finer counting of the number of terms.

Let us look at the condition in more detail.

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x &= t_0(x, y, z), & & \\
x &= t_h(x, y, x), & & \text{for } \mathbf{1} < h \le n, \\
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With respect to the Jónsson condition, here we exchange odd and even, so the condition resembles the ALVIN condition.

Henceforth t_1 satisfies $x = t_1(x, z, z)$ and $t_1(x, x, z) = t_2(x, x, z)$.

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Gumm terms can be seen in two ways.

Gumm terms as the "composition" of a Maltsev term with Jónsson terms (Gumm 1981).

If all the terms $t_2, \ldots, t_n$ are trivial projections onto the third coordinate, then t_1 satisfies the Maltsev condition for permutability.
On the other hand, if t_1 is the trivial projection onto the first coordinate, then the remaining terms are Jónsson terms (for $n-1$, shifting the indices).

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- (Gumm) *Congruence modularity is permutability composed with distributivity!*
- Indeed, Gumm Theorem and subsequent refinements show that any *reasonable* property which holds both in congruence permutable varieties and in congruence distributive varieties, holds in congruence modular varieties, as well.

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- In this way we can also appreciate the strength of Gumm condition: congruence modularity falls short of being equivalent to congruence distributivity just for a missing equation!
  This possibly explains the usefulness of Gumm terms. In a sense, it shows that modularity, though weaker, is not really far away from distributivity.

We now present the definition of directed Jónsson terms.

- THEOREM (Kazda, Kozik, McKenzie, Moore 2018) A variety $\mathcal{V}$ is congruence distributive if and only if there exist some $n \in \mathbb{N}$ and ternary terms $t_0, \dots, t_n$ such that

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Many applications; possibly the beginning of a completely new theory for congruence distributive varieties. (Even more general notions in Kazda, Valeriote 2020.)
Also a similar characterization of congruence modularity by means of *directed Gumm terms*.

Problems.

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  - Possibly, brand new ideas are needed. On the other hand,
  - We know the solutions with an approximation of at most 2 (due to the case n even, since we deal with isotonic properties).
  - The assumption n is even is not needed in the cases of near-unanimity and directed terms.

- We have got some positive results, generally in one of two possible directions.
  The converse problems seem much more difficult. In particular:
  - The original LTT problem is still untouched.
    How many Gumm terms do we get from a set of Day terms?
  - (Kazda, Kozik, McKenzie, Moore 2018) How many Jónsson directed terms do we get from a set of Jónsson terms?

Thank you!

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