Amalgamation for conic idempotent residuated lattices

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Residuated lattices

A *residuated lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, 1)$ is a monoid and
- for all $a, b, c \in A$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Examples/connections/applications:

1. Ideals of rings, under the usual multiplication and division of ideals. (Ward and Dilworth)
2. Lattice-ordered groups: $x\backslash y = x^{-1}y,\ y/x = yx^{-1}$.
3. Boolean and Heyting algebras, where $x \cdot y = x \wedge y$ (add a constant 0).
4. Relation algebras: $R\backslash S = (R^{\cup} \circ S^{c})^{c}, \quad S/R = (S^{c} \circ R^{\cup})^{c}$.
5. Mathematical linguistics: Context-free grammars, pregroups. (Lambek)
6. Non-classical logics: Linear, relevance, MV, BL, MTL, where multiplication is strong conjunction. Connections to proof theory.
7. CS: Action algebras.
8. CS: Memory allocation, pointer management, concurrent programming. (Separation logic, bunched implication logic).

Outline

1. Decompositions of conic idempotent
2. The structure of idempotent residuated chains
3. Strong amalgamation

Amalgamation

A class $\mathcal{K}$ of similar algebras has the *amalgamation property* if every *V-formation* in $\mathcal{K}$ (an ordered quintuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f_{\mathbf{B}}, f_{\mathbf{C}})$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $f_{\mathbf{B}} \colon \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} \colon \mathbf{A} \to \mathbf{C}$ are embeddings) has an *amalgam* $\mathcal{K}$ (an ordered triple $(\mathbf{D}, g_{\mathbf{B}}, g_{\mathbf{C}})$, where $\mathbf{D} \in \mathcal{K}$ and $g_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{D}$ and $g_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{D}$ are embeddings such that $g_{\mathbf{B}} \circ f_{\mathbf{B}} = g_{\mathbf{C}} \circ f_{\mathbf{C}}$). For classes closed under isomorphisms, we may assume that $f_{\mathbf{B}}$ and $f_{\mathbf{C}}$ are the inclusion maps (and $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $\mathbf{C}$).

The maps $g_{\mathbf{B}}$ and $g_{\mathbf{C}}$ can also be assumed to inclusions, but only if we identify/rename elements: $b \in \mathbf{B}$ and $c \in \mathbf{C}$ are identified when $g_{\mathbf{B}}(b) = g_{\mathbf{C}}(c)$. This becomes a mute point in the context of the *strong amalgamation property*: $g_{\mathbf{B}}[B] \cap g_{\mathbf{C}}[C] = g_{\mathbf{B}} \circ f_{\mathbf{B}}[A]$.

A class of similar algebras $\mathcal{K}$ closed under isomorphisms has the strong amalgamation property iff whenever algebras $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{K}$ intersect at a common subalgebra $\mathbf{A}$, there exists an algebra $\mathbf{D}$ in $\mathcal{K}$ having $\mathbf{B}$ and $\mathbf{C}$ as sublagebras.

Strong amalgamation $\Leftrightarrow$ Amalgamation + Epimorphism Surjectivity (for varieties)

Epimorphism Surjectivity $\Leftrightarrow$ Beth definability. Amalgamation $\Leftrightarrow$ Interpolation.

Residuated lattices

Part of the study of residuated lattices draws from ℓ-groups (characterizing congruences) and many classes of residuated lattices are related to ℓ-groups:

1. Commutative cancellative residuated lattices are conuclear images ℓ-groups.
2. (Generalized) MV-algebras are truncations of ℓ-groups.
3. BL-chains are ordinal sums of MV-algebras.

We will focus on the other end of the spectrum: idempotent multiplication.

A residuated lattice is called

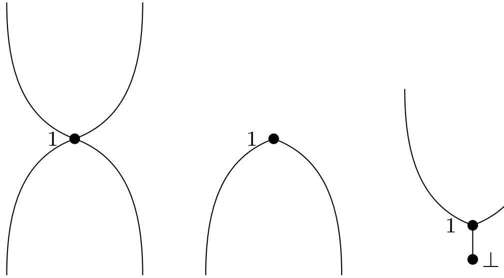
1. *commutative* if it satisfies $xy = yx$ (In this case we write $x \to y$ for $x\backslash y = y/x$.)
2. *integral* if it satisfies $x \leq 1$
3. *idempotent* if it satisfies $x^2 = x$
4. *conic* if every element is comparable to the monoid identity.

The elements of the *negative cone* $A^- = \{x \in A : x \leq 1\}$ have *negative sign* and the elements of the *positive cone* $A^+ = \{x \in A : x \geq 1\}$ have *positive sign*. A residuated lattice $\mathbf{A}$ is conic iff $A = A^- \cup A^+$.

Examples of commutative conic idempotent

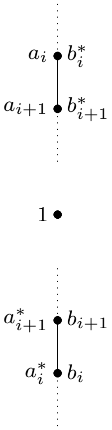
Heyting/Brouwerian algebras are (bounded) residuated lattices satisfying $xy = x \wedge y$ and are precisely the conic idempotent that are integral:

$$x \wedge y = (x \wedge y)(x \wedge y) \leq xy \leq x \cdot 1, 1 \cdot y \leq x \wedge y.$$



For positive elements x and y, we have $xy = x \vee y$.

Odd Sugihara monoids



The index set can be any chain, say $\mathbb{Z}$. The product of two elements is *the furthest away from* 1. For example, $b_0 a_1 = a_1 b_0 = b_0$.

If there is a tie, then it is the smallest: $b_0 a_0 = a_0 b_0 = b_0$.

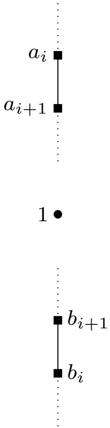
We define $x^* := x \to 1$. Distance from 1 can be defined by $|x| = x \vee x^*$ $(-|x| = x \wedge x^*)$.

So, inversion and the order determine the multiplication. Divisions split into multiplication and inversion.

The variety of odd Sugihara monoids is generated by these residuated chains. They are axiomatized as: 1-involutive ($x^{**} = x$), semilinear (generated by chains) and idempotent.

J. Raftery has studied commutative idempotent residuated structures (partly because of the connections to relevance logic), and was the one to define conic.

Non-commutative chains



In [G2004] there are examples of non-commutative idempotent residuated chains. The index set can be taken to be $\mathbb{Z}$ or $\mathbb{N}$ or a finite set. We are also given a subset J of the index set.

The product of two elements is defined to be *the furthest away from* 1. For example, $b_0 a_1 = a_1 b_0 = b_0$. If there is a tie, then

1. if $i \in J$, then $a_i b_i = a_i$ and $b_i a_i = b_i$ (Left), and
2. if $i \notin J$, then $a_i b_i = b_i$ and $b_i a_i = a_i$ (Right).

The operations $x^\ell := 1/x$ and $x^r := x\backslash 1$ (inverses) are defined and their behavior is investigated.

Divisions split: they are expressible via multiplication and inverses.

The non-commutative case

1. [G2004] Uncountably many examples of non-commutative idempotent chains. (Some generate equal varieties, but still we obtain uncountably many minimal varieties/atoms.)
2. [S2007] D. Stanovsky, basic facts of idempotent residuated lattices.
3. [CZ2009] W. Chen, X. Zhao. Idempotent residuated chains: via the natural order
4. [CZG2009] W. Chen, X. Zhao, X. Guo. Conic idempotent residuated lattices: via the natural order
5. [CC2019] W. Chen, Y. Chen. Conic idempotent residuated lattices: decomposition as ordinal sums of lattices
6. [GJM2020] J. Gil-Ferez, P. Jipsen, G. Metcalfe. Idempotent residuated chains: 1. finite, 2. commutative
7. [JTV] P. Jipsen, O. Tuyt, D. Valota. Finite involutive commutative idempotent residuated lattices.

Semigroup-theoretic approach

[CC2019] Conic idempotent residuated lattices also decompose as ordinal sums of lattices.

The arguments are given in a semigroup-theoretic terms, using Green's relations, in particular relation $\mathcal{D}$. (Unfortunately, it is not even mentioned what this relation boils down to.)

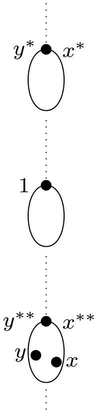
Many results are not presented in the way that is most usable. For example the block of a is defined to be:

$$C_a^A = \begin{cases} \{b \in A^+ : \forall c \in A^-, bc = c \Leftrightarrow ac = c \text{ and } cb = c \Leftrightarrow ca = a\} & a \in A^+ \\ \{b \in A^- : \forall c \in A^+, bc = c \Leftrightarrow ac = c \text{ and } cb = c \Leftrightarrow ca = a\} & a \in A^- \end{cases}$$

Subalgebra generation is not studied. (In particular, the role of inverses is ignored in the decomposition.)

Since understanding subalgebras is crucial in amalgamation, we need to rework the decomposition in a more illuminating way.

The role of the negation in commutative idempotent conic



In a commutative idempotent conic residuated lattice $\mathbf{A}$, we define $x^* := x \to 1$.

Note that $x \leq y^* \;\Leftrightarrow\; y \leq x^*$, so we have a Galois connection.

Therefore $x \mapsto x^{**}$ is a closure operator.

Actually, it is also a nucleus: $x^{**}y^{**} \leq (xy)^{**}$. Therefore, the image $\mathbf{A}^{**}$ is a residuated lattice (generally: same $\wedge$ and $\to$; adjusted $\vee$, $\cdot$ and 1).

The image of the nucleus is $A^* = \{a^* : a \in A\}$, it is a subalgebra of $\mathbf{A}$ and and it is 1-involutive. Hence $\mathbf{A}^*$ is an odd Sugihara monoid.

The blocks are of the form $\{x : x^{**} = a\}$, for values of $a \in A^{**}$.

Weak conservativity

$\mathbf{A}$ is called *conservative*, if $xy \in \{x, y\}$ for all $x, y \in A$. All conservative RLs are chains by [GJM2020].

Lemma. For a residuated lattice/ℓ-monoid $\mathbf{A}$ the following are equivalent.

1. $\mathbf{A}$ is conic and idempotent.
2. $\mathbf{A}$ is conic and *weakly conservative*: $xy \in \{x, y, x \wedge y, x \vee y\}$.
3. For all $x, y \in A$, we have $xy \in \{x \wedge y, x \vee y\}$.

In conic idempotent residuated lattices:

1. Conservativity for elements of different sign: For $x \leq 1 \leq y$, $xy \in \{x, y\}$.
2. Commutativity for elements of the same sign. (Proof as in Brouwerian case). For $x, y \leq 1$, $xy = x \wedge y$; for $x, y \geq 1$, $xy = x \vee y$.

Bringing the inverses back in the study

For an element x, we define its inverses $x^\ell := 1/x$ and $x^r := x\backslash 1$. They form a Galois connection, so $x \mapsto x^{r\ell}$ and $x \mapsto x^{\ell r}$ are closure operators, but not a nuclei.

$\mathbf{A}$ is called 1-*cyclic*, if $x^\ell = x^r$, $\forall x \in A$, iff $xy \leq 1 \Leftrightarrow yx \leq 1$, $\forall x, y \in A$.

Proposition. A conic idempotent residuated lattice is cyclic iff it is commutative.

Lemma. For every $x \in A$, x^ℓ and x^r are *conical* (comparable to every element).

We define $x^* := x^\ell \vee x^r$, $x^\star := x^\ell \wedge x^r$. So, $\{x^\ell \wedge x^r, x^\ell \vee x^r\} = \{x^r, x^\ell\}$.

Lemma. For every element a in an idempotent conic residuated lattice $\mathbf{A}$:

1. If a is central (equivalenty, $a^\ell = a^r$), then $aa^* = a \wedge a^*$.
2. If a is not central (equivalenty, $a^\ell \neq a^r$), then $\{a, a^*\}$ forms a left-zero or a right-zero semigroup and a commutes with all other elements.

Also, a^ℓ and a^r form a covering pair.

The set of inverses and closure operator

For an idempotent conic residuated lattice $\mathbf{A}$, we define the set of inverses:

$$A^i = \{a^\ell : a \in A\} \cup \{a^r : a \in A\}.$$

Note that A^i forms a chain.

We define the map γ on A by $\gamma(x) = x^{\ell r} \wedge x^{r\ell}$.

Lemma. If $\mathbf{A}$ is an idempotent conic residuated lattice, then

1. γ is a closure operator.
2. $\gamma[A] = \{x \in A : x = x^{\ell r} \wedge x^{r\ell}\} = A^i$.
3. The sets $\gamma^{-1}[\{a\}]$, where $a \in A^i$, form convex subposets of $\mathbf{A}$ with top element a, which are *prelattices*: adding a bottom yields a lattice. Also, they are ordered linearly according to the value of a.

Nucleus

A *nucleus* on a residuated lattice $\mathbf{A}$ is a closure operator γ on $\mathbf{A}$ satisfying

$$\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y).$$

Given a residuated lattice $\mathbf{A}$ and a nucleus γ on $\mathbf{A}$, the *nuclear image* of γ is the residuated lattice $\mathbf{A}_\gamma = (\gamma[A], \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1))$, where

$$x \cdot_\gamma y := \gamma(x \cdot y) \qquad x \vee_\gamma y := \gamma(x \vee y).$$

Theorem. If $\mathbf{A}$ is an idempotent conic residuated lattice, then

1. γ is a nucleus ($\gamma(x) = x^{\ell r} \wedge x^{r\ell}$).
2. A^i is the universe of a totally-ordered subalgebra $\mathbf{A}^i$ of $\mathbf{A}$, that further satisfies $x^{\ell r} \wedge x^{r\ell} = x$.

A residuated lattice is called *quasi-involutive* if it satisfies $x^{\ell r} \wedge x^{r\ell} = x$.

Key Lemma. Every strictly negative element of A^i is meet irreducible in $\mathbf{A}$.

Characterization of divisions

For $x, y \in A$, we have

$$x\backslash y = \begin{cases} x^r \vee y = \gamma(x)^r \vee y & x \leq y \\ x^r \wedge y = \gamma(x)^r \wedge y & \gamma(y) < \gamma(x), \text{ or } 1 < \gamma(x) = \gamma(y) \text{ and } x \not\leq y \\ x \Rightarrow y & \gamma(x) = \gamma(y) \leq 1 \text{ and } x \not\leq y. \end{cases}$$

$$y/x = \begin{cases} x^\ell \vee y = \gamma(x)^\ell \vee y & x \leq y \\ x^\ell \wedge y = \gamma(x)^\ell \wedge y & \gamma(y) < \gamma(x), \text{ or } 1 < \gamma(x) = \gamma(y) \text{ and } x \not\leq y \\ x \Rightarrow y & \gamma(x) = \gamma(y) \leq 1 \text{ and } x \not\leq y. \end{cases}$$

Here, $x \Rightarrow y$ is the Brouwerian implication within that block.

Decomposition

If $\mathbf{S}$ is a (quasi-involutive $x^{\ell r} \wedge x^{r\ell} = x$) idempotent residuated chain and for every $s \in S$, $\mathbf{A}_s$ is a prelattice with top element s where

1. $\mathbf{A}_s$ is a lattice, if s has no lower cover in S.
2. $\mathbf{A}_s$ is a Brouwerian algebra, if s negative.
3. If s is not central, then A_s is trivial.

then $(\mathbf{S}, \{\mathbf{A}_s : s \in S\})$ is called a *decomposition system*.

Given such a decmposition system $D = (\mathbf{S}, \{\mathbf{A}_s : s \in S\})$, we consider the ordinal sum $A_D := \bigoplus_{s \in S} A_s$ and define the multiplication and divisions using the formulas from the previous slides.

Theorem. Given a decomposition system and conic idempotent residuated lattices are in bijective correspondence. Actually, this gives rise to a categorical equivalence.

Amalgamation: strategy

To strongly amalgamate conic idempotent residuated lattices, first strongly amalgamate:

1. The skeletons: quasi-involutive idempotent residuated chains (com: odd Sugihara chains)
2. the negative blocks: Brouwerian algebras
3. the positive blocks: topped prelattices

For (2) and (3) we have even the *super-amalgamation property*:

$$b \leq^{\mathbf{D}} c \text{ iff } \exists a \in A, b \leq^{\mathbf{B}} a \leq^{\mathbf{C}} c.$$

Idempotent residuated chains

In every idempotent residuated chain we have

$$xy = \begin{cases} x & y \in (x^r, x] \text{ or } y \in [x, x^r] \\ y & x \in (y^\ell, y] \text{ or } x \in [y, y^\ell] \end{cases}$$

$$x\backslash y = \begin{cases} x^r \vee y & x \leq y \\ x^r \wedge y & y < x \end{cases}$$

$$y/x = \begin{cases} x^\ell \vee y & x \leq y \\ x^\ell \wedge y & y < x \end{cases}$$

These equations define the idempotent residuated chain from its $\{\wedge, \vee, 1, {}^r, {}^\ell\}$-reduct.

Reducts if idempotent residuated chains

An algebra $(A, \wedge, \vee, {}^{\ell}, {}^{r}, 1)$ is called an *idempotent Galois connection* if it satisfies the following positive universal sentences:

1. $(A, \wedge, \vee)$ is a lattice,
2. $x \leq y$ or $y \leq x$ (totally ordered),
3. $x \leq x^{\ell r}$, $x \leq x^{r\ell}$ and $(x \vee y)^{\ell} = x^{\ell} \wedge x^{r}$ (Galois connection on a lattice),
4. $y \leq x^{r} \wedge x^{\ell}$ or $x^{r} \vee x^{\ell} \leq y$ (there is no element between x^{ℓ} and x^{r}),
5. $1^{\ell} = 1^{r} = 1$.

Theorem. Idempotent residuated chains are definitionally equivalent to idempotent Galois connection, which are their reducts.

Condition 4 is crucial in establishing associativity of multiplication.

Flow diagrams

Let a be a positive and b a negative element of an idempotent residuated chain $\mathbf{A}$.

$a\ L\ b$ means that $\{a, b\}$ forms a left-zero semigroup.
$a\ R\ b$ means that $\{a, b\}$ forms a right-zero semigroup.

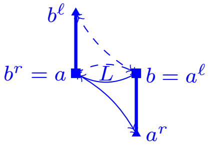
Corollary. Let $\mathbf{A}$ be an idempotent residuated chain.

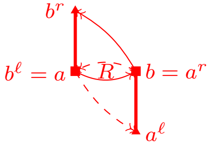
If a is a positive non-central element of $\mathbf{A}$, then exactly one of the following situations happen.

1. $a^{\ell\ell} \prec a^{\ell r} = a\ L\ a^{\ell} = a^{*} \succ a^{r}$.
2. $a^{rr} \prec a^{r\ell} = a\ R\ a^{r} = a^{*} \succ a^{\ell}$.

If b is a negative non-central element of $\mathbf{A}$, then exactly one of the following situations happen.

1. $b^{\ell} \prec b^{*} = b^{r}\ L\ b = b^{r\ell} \succ b^{rr}$.
2. $b^{r} \prec b^{*} = b^{\ell}\ R\ b = b^{\ell r} \succ b^{\ell\ell}$.



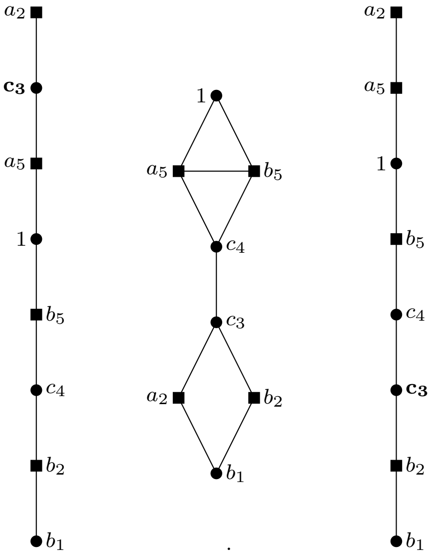


Two algebras with the same monoidal preorder

The *monoidal preorder* [GJM2020], is defined by: $x \sqsubseteq y$ iff $xy = x$.

It encodes the multiplication operation, but there are different idempotent residuated structures on the same set that have the same monoidal preorder.

Recall that Hasse diagrams for preordered sets have horizontal line segments connecting mutually comparable elements.



Enhanced monoidal preorder

Given an idempotent residuated chain $\mathbf{A}$, we define its enhanced monoidal preorder to consist of the following

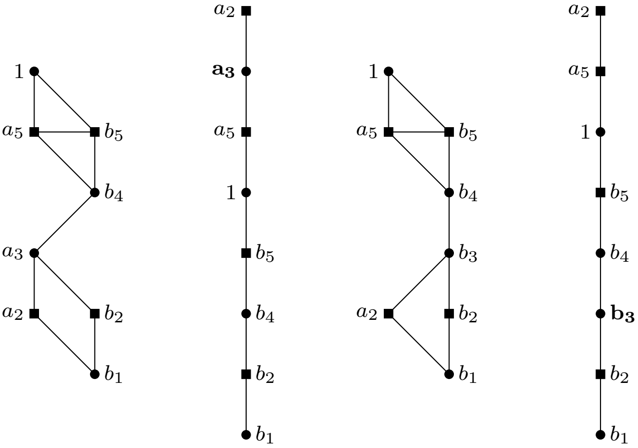
1. the monoidal preorder $(A, \sqsubseteq)$, as defined earlier,
2. the positive cone A^+ and is the negative cone A^-, and
3. for $a \in A$, $a^\star = a^\ell \wedge a^r$.

These properties lead to the following definition: $(\mathbf{P}, P^+, P^-, {}^\star, 1)$ is an *enhanced monoidal preorder*, if $\mathbf{P} = (P, \sqsubseteq)$ is a pre-ordered set of width of less than or equal to two, with maximum element 1, P^+ and P^- are totally-ordered subsets such that $P^+ \cup P^- = P$ and $P^+ \cap P^- = \{1\}$, $1^\star = 1$ and

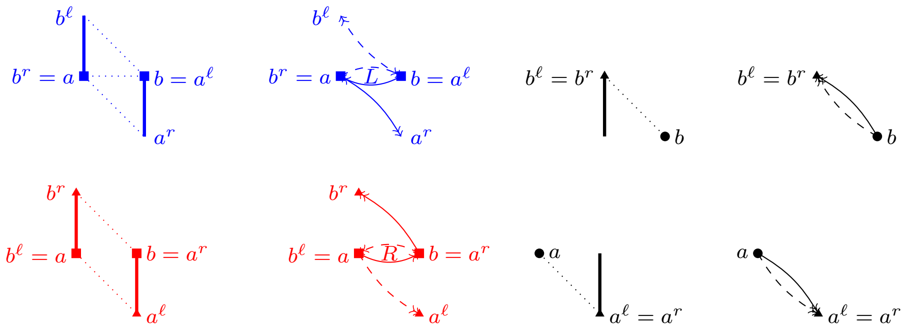
1. for $b \in P^-$, $b^\star$ is the smallest element of P^+ such that $b \sqsubset b^\star$
2. for $a \in P^+$, $a^\star$ is the largest element of P^- such that $a^\star \sqsubset a$.
3. the preordered is *layered*: if two distinct elements are not related by $\sqsubset$ nor $\sqsupset$, then they have different signs and their $\sqsubset$-upsets and downsets coincide.

Since the width is two, in Hasse diagrams of enhanced monoidal preorders we use a left column for positive elements and a right column for negative elements.

Distinguishing the algebras with the same monoidal order



Enhanced monoidal-preorders and the flow diagrams



$a \dashrightarrow a^\ell$ ● central

$a \longrightarrow a^r$ ▲ any

$a \overset{a \prec b}{\textbf{———}} b$ ■ non-central

Correspondence

Conversely, given an enhanced monoidal preorder. We define

1. $A = P$,
2. $xy = yx = x$, if $x \sqsubset y$ or $x = y$
3. $xy = x$ and $yx = y$, if $x \sqsubseteq y$ and $y \sqsubseteq x$
4. $xy = y$ and and $yx = x$, if x and y are incomparable
5. $x \leq y$, if $(x, y \in P^-$ and $x \sqsubseteq y)$ or $(x, y \in P^+$ and $y \sqsubseteq x)$
6. x^ℓ and x^r according to the flow diagrams.

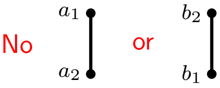
Theorem. Idempotent residuated chains are in bijective correspondence to enhanced monoidal preorders.

We make use of idempotent Galois connections as an intermediate step.

Lemma. Let $\mathbf{A}$ and $\mathbf{B}$ be idempotent residuated chains and let $\mathbf{P_A}$ and $\mathbf{P_B}$ be the corresponding enhanced monoidal preorders. Then $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ iff $\mathbf{P_A}$ is closed under same-level elements, under * and it contains 1.

Connected components

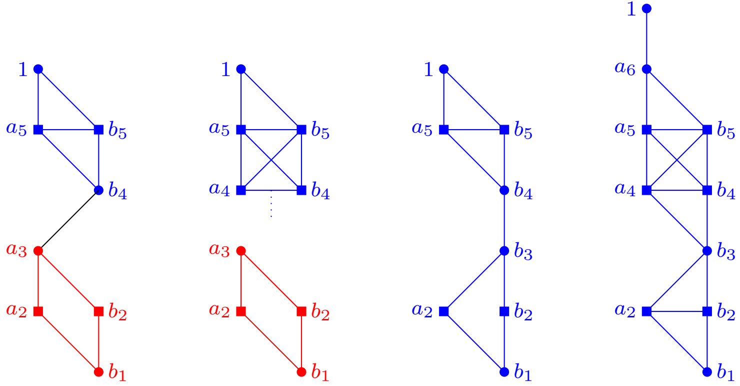
Theorem. The enhanced monoidal preordered sets of quasi-involutive idempotent chains are precisely the ones for which there are no covering pairs of central elements of the same sign.



We say that a is *connected* to b if (a, b) is in the equivalence relation relation generated by the set $\{(x, y) : x^\ell = y \text{ or } x^r = y\}$. The equivalence classes are called *connected components* and they are the maximal union-indecomposable 1-free subalgebras.

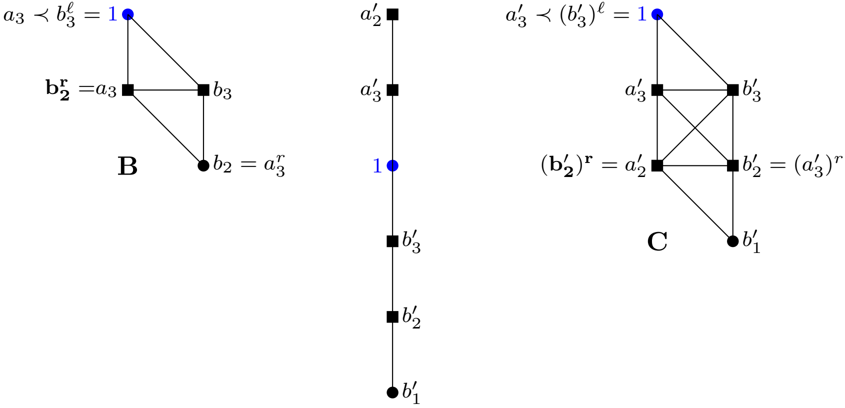
Lemma. A subset Q of an enhanced monoidal preorder is a connected component iff it it is closed under same-level elements and $a \in Q \Leftrightarrow a^\star \in Q$. (Then Q is also level-convex.)

Conected components of enhanced monoidal preorders



Failure of amalgamation for (conic) idempotent chains

Here $A = \{1\}$, $\mathbf{B}$ is on the left, $\mathbf{C}$ is on the right and they represent quasi-involutive idempotent residuated chains.



Rigidity

A subset is *strongly connected* if between any two non-identity elements there is a directly path of applications of $^\ell$ and ^{r}. We want connected components to be strongly connected.

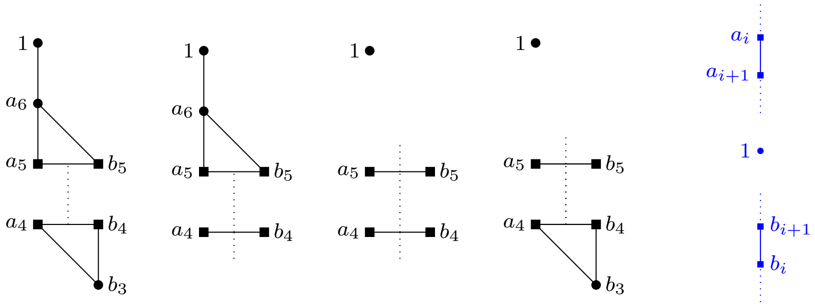
An residuated lattice is called *rigid* if it satisfies $x^r = x^{r\star\star}$ and $x^\ell = x^{\ell\star\star}$.

Lemma.

1. A conic idempotent residuated lattice is rigid iff its quasi-involutive skeleton is rigid.
2. A quasi-involutive idempotent residuated chain is rigid iff it is *-involutive iff its connected components are strongly connected.

Star-involutive residuated chains

Lemma. The connected *-involutive residuated chains are exactly *crowns* (their enhaned monoidal preorder is a vertical crown) and are exactly the chains in [G2004] (the horizontal lines are optional).

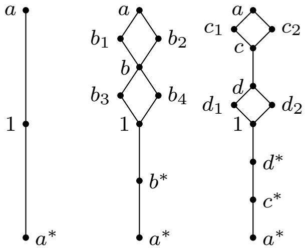


Corollary. The *-involutive residuated chains are ordinal/nested sums of crowns. Subalgebras are unions of connected components.

Theorem. *-involutive residuated chains have the strong amalgamation property.

Failure of strong amalgamation for rigid commutative conic idempotent

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be the commutative conic idempotent residuated lattices with rigid skeletons $\{a^*, b^*, 1, b, a\}$ and $\{a^*, c^*, d^*, 1, d, c, a\}$.



In any amalgam $\mathbf{D}$, b is an inverse and so is c, hence they are conical and thus comparable. If $b < c$ in $\mathbf{D}$, then since $b_1 \wedge b_2 = b$ and since c is conical, we would get $b_1, b_2 \leq c$, hence $a = b_1 \vee b_2 \leq c$, a contradiction, so $c \leq b$. Likewise, $b \leq c$ and also $b = d$, this $c = d$.

Strong amalgamation for De Morgan rigid conic idempotent

More generally, the issue is that the positive blocks are not lattices and the meet of elements of the same block is the top element of the next block below.

Lemma. In a conic idempotent residuated lattice all blocks are lattices iff it satisfies $\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y)$ (*De Morgan*).

Theorem. The class of *rigid De Morgan* conic idempotent residuated lattices has the strong amalgamation property.

Strong amalgamation extended

Combining [Metcalfe-Montagna-Tsinakis, J. of Algebra 2010] and [Campercholi, J. Symb. Log. 2018], we can prove:

Theorem. Let $\mathcal{V}$ be an arithmetical variety whose finitely subdirectly irreducible members form a universal class. Let $\mathcal{S}$ be a subclass of $\mathcal{V}$ where:

1. $\mathcal{S}$ contains all finitely subdirectly irreducible member of $\mathcal{V}$;
2. $\mathcal{S}$ is closed under isomorphisms and subalgebras;
3. Each V-formation consisting of algebras in $\mathcal{S}$ has a strong amalgam in $\mathcal{V}$;
4. For any algebra $\mathbf{B} \in \mathcal{V}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if Θ is a congruence of $\mathbf{A}$ and $\mathbf{A}/\Theta \in \mathcal{S}$, then there exists a congruence Ψ of $\mathbf{B}$ such that $\Psi \cap A^2 = \Theta$ and $\mathbf{B}/\Psi \in \mathcal{S}$.

Then $\mathcal{V}$ has the strong amalgamation property.

We take $\mathcal{V}$: rigid De Morgan semiconic idempotent residuated lattices and $\mathcal{S}$: finitely subdirectly irreducible in $\mathcal{V}$ (i.e., 1 is join-irreducible).

Condition (4) is a version of the Congruence Extension Property:
For any algebra $\mathbf{B} \in \mathcal{V}$ and subalgebra $\mathbf{A}$ of $\mathbf{B}$, if Θ is a congruence of $\mathbf{A}$, then there exists a congruence Ψ of $\mathbf{B}$ such that $\Psi \cap A^2 = \Theta$.

Congruence filters

Given a residuated lattice $\mathbf{A}$, congruences on $\mathbf{A}$ are in bijective correpondence to congruence filters (aka deductive filters):

subsets F such that

1. F is a filter
2. F is submonoid
3. if $a \in F$ and $x \in A$, then $x\backslash ax,\ xa/x \in F$. (closed under conjugation)

The congruence filter associated to a congruence θ is $F_\theta = \uparrow[1]_\theta$.

The congruence associated to a filter F is given by: $x\ \theta_F\ y$ iff $x\backslash y,\ y\backslash x \in F$.

Congruences also correspond to *convex normal* $(x\backslash ax \wedge 1,\ xa/x \wedge 1)$ *subalgebras*: $[1]_\theta$, as well as to *convex normal submonoids* of negative elements $[1]_\theta^-$.

Congruence generation

Lemma. Let $\mathbf{A}$ be a semiconic idempotent residuated lattice and $Y \subseteq A$.

1. $\mathbf{A}$ satisfies the identities $y \wedge y^{\ell\ell} \wedge y^{rr} \leq xy/x$ and $y \wedge y^{\ell\ell} \wedge y^{rr} \leq x\backslash yx$.
2. The congruence filter of A generated by Y is given by $\langle Y \rangle = \uparrow\{s_{n_1}(y_1) \wedge \cdots \wedge s_{n_k}(y_k) : k, n_1, \ldots, n_k \in \mathbb{N}, y_1, \ldots, y_k \in Y \wedge 1\}$, where $s_n(y) := y \wedge y^{\ell\ell} \wedge y^{rr} \wedge y^{\ell\ell rr} \wedge \cdots \wedge y^{\ell^{2n}} \wedge y^{r^{2n}}$.

Lemma. If Y is a congruence filter and $a \in A$, then

$$\langle Y \cup \{a\} \rangle = \uparrow\{y \wedge s_n(a) : n \in \mathbb{N}, y \in Y\},$$

Recall that (in any variety of residuated lattices, say) the CEP is equivalent to the existence of a local deduction theorem for the corresponding logic and to the demand that congruence generation is based on ideal terms *without paremeters*.

Theorem. The variety of semiconic idempotent residuated lattices has the CEP.

Corollary. The variety of rigid De Morgan semiconic idempotent residuated lattices has the CEP.

Proof of condition (4)

Let F be the congruence filter of $\mathbf{A}$ corresponding to Θ.

By the CEP, $F_B :=\uparrow F \cap A$ is a congruence filter of B with $F_B \cap A = F$.

By Zorn's Lemma, let G be a maximal such. We will show that $[1_{\mathbf{B}}]_G$ is $\vee$-irr.

For $x, y \in B^-$ with $[x]_G \neq [1]_G$ and $[y]_G \neq [1]_G$ (i.e., $x \notin G$ and $y \notin G$), we will show that $[x]_G \vee [y]_G \neq [1]_G$. BWOC, assume $[x]_G \vee [y]_G = [1]_G$.

By maximality of G, $a \in \langle G \cup \{x\} \rangle$ and $b \in \langle G \cup \{y\} \rangle$, for some $a, b \in A^- \setminus F$.

So, $g_x \wedge s_m(x) \leq a$ and $g_y \wedge s_n(y) \leq b$, for some $g_x, g_y \in G^-$ and $m, n \in \mathbb{N}$.

So we have (the first equality still to be shown on next slide)

$$[1]_G = [s_m(x)]_G \vee [s_n(y)]_G = [g_x \wedge s_m(x)]_G \vee [g_y \wedge s_n(y)]_G \leq [a]_G \vee [b].$$

Therefore, $a \vee b \in G \cap A = F$, so $[1]_F \leq [a]_F \vee [b]_F$, contradicting that $\mathbf{A}/F$ is FSI (and that $[1]^F$ is $\vee$-irr).

Axiomatization of idempotent semiconic: simplified

We follow the general process given in [G2003]

$1 \leq x$ or $x \leq 1$	(positive universal formula
$1 \leq x$ or $1 \leq x\backslash 1$	axiomatizing
$1 = x \wedge 1$ or $1 = (x\backslash 1) \wedge 1$	the class of conic)
$1 = \gamma_1(x \wedge 1) \vee \gamma_2(x^r \wedge 1)$	(equations axiomatizing the variety of semiconic)

where γ_1 and γ_2 are *iterated conjugates*: arbitrary compositions of the polynomials $\lambda_a(x) = a\backslash xa \wedge 1$ and $\rho_b(x) = bx/b \wedge 1$, for various values of a, b.

Lemma. Semiconic idempotent residuated lattices satisfy the implications:

1. $x \vee y = 1 \Rightarrow \lambda_z(x) \vee y = 1$.
2. $x \vee y = 1 \Rightarrow x \vee \rho_w(y) = 1$.
3. $x \vee y = 1 \Rightarrow \gamma_1(x) \vee \gamma_2(y) = 1$, for all iterated conjugates γ_1 and γ_2.
4. $x \vee y = 1 \Rightarrow s_n(x) \vee s_m(y) = 1$, for all $n, m \in \mathbb{N}$.

Theorem. Idempotent semiconic residuated lattices are axiomatized by the 1-variable identities: $x^2 = x$ and

$$(x \wedge x^{\ell\ell} \wedge x^{rr} \wedge 1) \vee (x^r \wedge x^{\ell} \wedge x^{rrr} \wedge 1) = 1.$$