Hyperprincipal Generators for Regular and Good Ultrafilters

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PALS

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Preliminary Definitions

Definition ($\mathcal{U}(A)$)

Let A be any set. The universe over A is the collection

$$\bigcup_{n\in\mathbb{N}} \mathcal{U}_n(A)$$

where $\mathcal{U}_0(A) := A$ and $\mathcal{U}_{n+1}(A) := \mathcal{P}(\mathcal{U}_n(A)) \cup \mathcal{U}_n(A)$.

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Every finitary function and relation on A appears in $\mathcal{U}(A)$, so every finitary structure on A can be thought of as a subset of $\mathcal{U}(A)$.

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Definition (Nonstandard Framework)

Given a set A, a nonstandard framework on A is a set ${}^*A$ and a function $*: \mathcal{U}(A) \to \mathcal{U}({}^*A)$ satisfying

1. $(* \upharpoonright A)$ *is an injective map* $A \hookrightarrow {}^*A$ *and* $*(A) = {}^*A$.

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1. $(* \restriction A)$ *is an injective map* $A \hookrightarrow {}^*A$ *and* $*(A) = {}^*A$.
2. ${}^*\emptyset = \emptyset$.
3. *(Transfer) For every bounded quantifier sentence φ in the language of set theory expanded by constants from $\mathcal{U}(A)$ we have that* $(\mathcal{U}(A) \vDash \varphi) \iff ({}^*\mathcal{U}(A) \vDash \varphi)$.

Note: ${}^*\mathcal{U}(A)$ is the substructure of $\mathcal{U}({}^*A)$ generated by the $\in$-downward closure of $*[\mathcal{U}(A)]$.

Preliminary Definitions

Definition (Enlargement)

If a nonstandard framework $* : \mathcal{U}(A) \to \mathcal{U}({}^*A)$ *also satisfies "for every* $F \in \mathcal{U}(A)$ *that is a collection of elements of rank greater than 0 with the finite intersection property (FIP) there is some*

$$b \in \bigcap_{f \in F} {}^*f$$

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with $b \in {}^*\mathcal{U}(A)$", we say that $*$ (or ${}^*\mathcal{U}(A)$) is an enlargement of $\mathcal{U}(A)$.

If A is the base set for a finitary structure $\mathcal{A}$ and $\Phi(x)$ is a type over A, the set $F = \{\varphi(\mathcal{A}) : \varphi(x) \in \Phi(x)\}$ has the FIP, and the guaranteed element b is a realization of the type $\Phi(x)$ in ${}^*A$.

Existence of Enlargements

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Theorem (Keisler)

Every enlargement is locally an ultrapower.

Hyperprincipal Generators

1. Suppose that $I \subseteq A$, $\mathcal{U}$ is an ultrafilter on I, and $*\colon \mathcal{U}(A) \to \mathcal{U}({}^*A)$ is a nonstandard framework. Then

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2. Suppose that $b \in {}^*\mathcal{U}(A)$ is such an element. Then $u(b) := \{B \subseteq I : b \in {}^*B\}$ has the FIP: for any finite $\Gamma \subseteq u(b)$ the statements

$$\exists i \in I, \bigwedge_{B \in \Gamma} i \in B \quad \text{and} \quad \exists i \in {}^*I, \bigwedge_{B \in \Gamma} i \in {}^*B$$

are equivalent by transfer, and the latter is witnessed by b for any Γ.

Hyperprincipal Generators

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4. Essentially the same construction works for filters in general (single element generators are replaced with generating sets), giving a Galois correspondence between $\mathcal{P}(\mathcal{P}(I))$ and $\mathcal{P}({}^*I)$ where the closed sets in $\mathcal{P}(\mathcal{P}(I))$ are filters on I.

Ultrapowers Using Generators

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2. Then $B^{u(i)}$ embeds in ${}^*A$ via $[f] \mapsto {}^*f(i)$.
3. For every function g and relation E in the structure, ${}^*g$ and ${}^*E$ restricted to the image of the above embedding are the corresponding functions and relations on the ultrapower.

Properties of $u(b) = \mathcal{U}$ from Properties of b

Question: How are the properties of an ultrafilter $u(b) = \mathcal{U}$ reflected in the properties of b?

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Proof: ($\Rightarrow$) If $b \in *[I]$ then $b = {}^*i$ for some $i \in I$. Transfer of ${}^*i \in {}^*B$ for each $B \in \mathcal{U}$ guarantees $i \in \bigcap \mathcal{U}$, so i is a generator for $\mathcal{U}$.

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$(\Leftarrow)$ If $\mathcal{U}$ is principal, $\{i\} \in \mathcal{U}$ for some $i \in I$. However, ${}^*\{i\} = \{{}^*i\}$ so the only possible hyperprincipal generator for $\mathcal{U}$ is ${}^*i$.

Existence of Nonstandard Elements

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2. The intersection $\bigcap_{i \in I} {}^*(I \setminus \{i\}) = \bigcap_{i \in I} {}^*I \setminus \{{}^*i\}$ does not contain any element of $*[I]$, but is inhabited, such elements generate nonprincipal ultrafilters.

What is Keisler's Order?

Theorem (Keisler, 1967)

If $\mathcal{L}$ is a countable language, $\mathcal{U}$ is a regular ultrafilter on I, and $\mathcal{A} \equiv \mathcal{B}$ are elementary equivalent $\mathcal{L}$-structures, then the ultrapower $\mathcal{A}^{\mathcal{U}}$ is $|I|^+$-saturated if and only if the ultrapower $\mathcal{B}^{\mathcal{U}}$ is $|I|^+$-saturated.

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Keisler's order is a pre-order on first-order countable theories defined by $T_1 \leq T_2$ iff for every index set I, every ultrafilter $\mathcal{U}$ on I, and every (or any!) $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$ we have $\mathcal{M}_1^{\mathcal{U}}$ is $|I|^+$-saturated implies $\mathcal{M}_2^{\mathcal{U}}$ is $|I|^+$-saturated.

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4. Keisler's order is not well-founded (Malliaris and Shelah, 2018) and has a continuum sized antichain (Malliaris and Shelah, 2021).
5. Keisler's order has a maximum class (Keisler, 1967) characterized by theories that are only saturated by good ultrafilters.

Regular Ultrafilters

Definition (Regular Ultrafilter)

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Theorem

If A is an infinite set and $\mathcal{U}$ an ultrafilter on I, then $\mathcal{U}$ is regular if and only if every $C \subseteq A^{\mathcal{U}}$ with $|C| \leq |I|$ is contained in an ultraproduct of the form $\prod_{i \in \mathcal{U}} B_i$ where each B_i is a finite subset of A.*

What is a Regular Hyperprincipal Generator?

Theorem (Regular Generator)

The ultrafilter $u(i)$ is regular on I if and only if for every (equivalently, any) $X \subseteq I$ with $|X| = |I|$ there is a function $f : I \to \mathcal{P}_\omega(I)$ such that $*[X] \subseteq {}^*f(i)$.

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For any set B in $\mathcal{U}(A)$ there is embedded $B^{u(i)} \to \mathcal{U}({}^*A)$ defined by $[f] \mapsto {}^*f(i)$, so $*[X] \subseteq {}^*f(i) \in {}^*\mathcal{P}_\omega(I)$ expresses that $*[X]$ "appears to be finite" in ${}^*\mathcal{U}(A)$.

Well-definedness of the Keisler Order

A sketch of a proof of the well-definedness of the Keisler order:

1. For a "small" type $\Phi(x)$ in a regular ultrapower (embedded in ${}^*\mathcal{U}(A)$), we can embed $\Phi(x)$ into a "hyperfinite" set of formula.

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2. A logically equivalent hyperfinite set of formula can be found in the ultrapower of any elementary equivalent structure by using transfer.
3. If the formulas corresponding to the original type are realized in our new structure, they must also be realized in the original structure.

Regular and Good Ultrafilters

Definition

An ultrafilter $\mathcal{U}$ on I is good if for every monotone $f : \mathcal{P}_\omega(I) \to \mathcal{U}$ there is a multiplicative function $g : \mathcal{P}_\omega(I) \to \mathcal{U}$ that is pointwise a subset of f.

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Theorem

An ultrafilter $\mathcal{U}$ on I is regular and good if and only if for every structure $\mathcal{B}$ in a countable language the ultrapower $\mathcal{B}^{\mathcal{U}}$ is $|I|^+$-saturated.

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Theorem

The ultrafilter $u(i)$ is both regular and good if and only if whenever F is a collection of functions $f : I \to \mathcal{U}(A)$ of bounded rank with $|F| \leq |I|$ and $B = \{{}^*f(i) : f \in F\}$ has the FIP there is a function $g : I \to \mathcal{U}(A)$ of bounded rank such that ${}^*g(i) \in \bigcap B$.

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Sketch of proof that regular and good ultrafilters saturate all theories: Take B to be the collection of sets $\varphi(\mathcal{C}^{u(i)})$ for each $\varphi(x)$ in the type.

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Sketch of proof that regular and good ultrafilters saturate all theories: Take B to be the collection of sets $\varphi(\mathcal{C}^{u(i)})$ for each $\varphi(x)$ in the type. B has the FIP by compactness, and g can be taken such that $g : I \to C$, so ${}^*g(i) \in \mathcal{C}^{u(i)}$.

Thank you!

Any Questions?