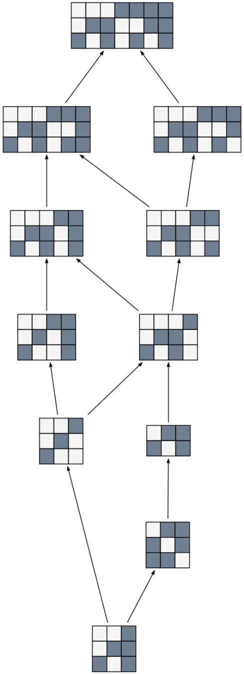
A classification of left exact categories

(motivated from universal algebra)

Michael Hoefnagel



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The matrix taxonomy of left exact categories

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Abstract

This paper is concerned with the problem of classifying left exact categories according to their 'matrix properties' — a particular category-theoretic property represented by integer matrices. We obtain an algorithm for deciding whether a conjunction of these matrix properties follows from another. Computer implementation of this algorithm allows one to peer into the complex structure of the poset of all 'matrix classes', i.e., the poset of all collections of left exact categories determined by these matrix properties. Among elements of this poset are the collections of Mal'tsev categories, majority categories, (left exact) arithmetical categories, as well as left exact extensions of various classes of varieties of universal algebras, obtained through a process of 'syntactical refinement' of familiar Mal'tsev conditions studied in universal algebra.

Talk overview

1. **Matrix properties**
   - What is a matrix property?
   - Left exact categories
   - Mal'tsev categories
   - Majority categories
   - Arithmetical categories

2. **Deciding implications of matrix properties**
   - Matrix classes
   - The algorithm
   - Some computer aided results
   - Context sensitivity
   - Future questions

Matrix properties in varieties

Consider a Mal'tsev condition which asserts the existence of an n-ary term p satisfying some equations of the form $p(x_1, x_2, \dots, x_n) = y$.

Matrix properties in varieties

Consider a Mal'tsev condition which asserts the existence of an n-ary term p satisfying some equations of the form $p(x_1, x_2, \ldots, x_n) = y$. Such a Mal'tsev condition may always be equivalently presented by finitely many such equations:

$$\begin{aligned} p(x_{11}, \ldots, x_{1m}) &= y_1, \\ &\vdots \\ p(x_{n1}, \ldots, x_{nm}) &= y_n. \end{aligned}$$

Such a condition may be represented a matrix:

$$\left[\begin{array}{ccc|c} x_{11} & \ldots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \ldots & x_{nm} & y_n \end{array}\right]$$

Examples

Many familiar Mal'tsev conditions are captured by such matrices:

- The matrix $\left[\begin{array}{ccc|c} x & x & y & y \\ y & x & x & y \end{array}\right]$ corresponds to a Mal'tsev term.
- The matrix $\left[\begin{array}{ccc|c} x & x & y & x \\ x & y & x & x \\ y & x & x & x \end{array}\right]$ corresponds to a majority term.
- The matrix $\left[\begin{array}{ccc|c} x & x & y & y \\ x & y & x & x \\ y & x & x & y \end{array}\right]$ corresponds to a Pixley term.

Closedness properties of internal relations

Matrix properties are formulated within the internal language of an abstract category, and formally these categorical properties are called *closedness properties of internal relations*:

- Z. Janelidze. Closedness properties of internal relations I: a unified approach to Mal'tsev, unital and subtractive categories. Theory and Applications of Categories, 16(12):236–261, 2006.

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Remark

Actually, matrix properties capture more Mal'tsev conditions than those mentioned above.

Matrix properties categorically

Let

$$M = \left[\begin{array}{ccc|c} x_{11} & \dots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \dots & x_{nm} & y_n \end{array}\right]$$

be a matrix where $x_{i1}, \dots, x_{im}, y_i \in X_i$. Given a family of sets $\overline{A} = (A_1, A_2, \dots, A_n)$, a row-wise interpretation of M of type $\overline{A}$ is any matrix of the form

$$M' = \left[\begin{array}{ccc|c} f_1(x_{11}) & \dots & f_1(x_{1m}) & f_1(y_1) \\ \vdots & & \vdots & \vdots \\ f_n(x_{n1}) & \dots & f_n(x_{nm}) & f_n(y_n) \end{array}\right]$$

where $f_i : X_i \to A_i$ are any maps.

M-closedness

A relation $R \subseteq A_1 \times \cdots \times A_n$ is said to be *strictly M-closed* if for any row-wise interpretation M' of M of type $(A_1, \ldots, A_n)$ given by

$$M' = \left[\begin{array}{ccc|c} x'_{11} & \cdots & x'_{1m} & y'_1 \\ \vdots & & \vdots & \vdots \\ x'_{n1} & \cdots & x'_{nm} & y'_n \end{array}\right]$$

the implication

$$\begin{bmatrix} x'_{11} \\ \vdots \\ x'_{n1} \end{bmatrix} \in R, \ldots, \begin{bmatrix} x'_{1m} \\ \vdots \\ x'_{nm} \end{bmatrix} \in R \implies \begin{bmatrix} y'_1 \\ \vdots \\ y'_n \end{bmatrix} \in R$$

holds.

The Mal'tsev matrix property

Definition (Riguet)

A relation $R \subseteq X \times Y$ is called *difunctional* if it satisfies

$$\begin{bmatrix} u \\ y \end{bmatrix} \in R \quad \text{and} \quad \begin{bmatrix} u \\ v \end{bmatrix} \in R \quad \text{and} \quad \begin{bmatrix} x \\ v \end{bmatrix} \in R \implies \begin{bmatrix} x \\ y \end{bmatrix} \in R,$$

for any $x, u \in X$ and $y, v \in Y$.

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for any $x, u \in X$ and $y, v \in Y$.

As a closedness property of relations

In other words, R is difunctional if and only if R is strictly closed with respect to the Mal'tsev matrix:

$$\left[\begin{array}{ccc|c} x & x & y & y \\ y & x & x & y \end{array}\right]$$

Mal'tsev varieties

Theorem

For a variety $\mathbb{V}$ the following are equivalent.

- *$\mathbb{V}$ admits a Mal'tsev term.*
- *The composite of congruences in $\mathbb{V}$ is commutative.*
- *Every reflexive compatible relation on an algebra in $\mathbb{V}$ is a congruence.*
- *Every compatible binary relation in $\mathbb{V}$ is difunctional.*

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- *Every compatible binary relation in $\mathbb{V}$ is difunctional.*

Remark

The last statement can be expressed purely categorically, using the notion of internal binary relation in a category.

Pullbacks

Recall that if $f : X \to Z$ and $g : Y \to Z$ are any morphisms in a category $\mathbb{C}$, a pullback of f along g is a commutative square

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

such that for any $\alpha : A \to X$ and $\beta : A \to Y$ with $f \circ \alpha = g \circ \beta$ there exists a morphism $\sigma : A \to P$ with $p_1 \circ \sigma = \beta$ and $p_2 \circ \sigma = \alpha$.

$$
\begin{tikzcd}
A \arrow[dr, dotted, "\sigma"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

Pullbacks in varieties

$$\begin{tikzcd}
\{(x,y) \mid f(x) = g(y)\} \arrow[r, "(x,y)\mapsto x"] \arrow[d, "(x,y)\mapsto y"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Left exact categories

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A category $\mathbb{C}$ is called left exact (or finitely complete) if it has all finite limits.

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Example

Top, **Pos**, **Grph**, any (quasi)variety of algebras, or the dual of any of these.

Internal relations in categories

Suppose that $\mathbb{C}$ is a category with binary products, recall that an *internal binary relation* R between objects X and Y in $\mathbb{C}$ is simply a monomorphism $r = (r_1, r_2) : R_0 \to X \times Y$.

Working set-theoretically in categories

Given morphisms $x : S \to X$ and $y : S \to Y$ we can write xRy or $(x, y) \in_S R$ if the dotted arrow exists making the diagram below commute.

$$\begin{tikzcd} & R_0 \arrow[d, "r"] \\ S \arrow[ur, dotted] \arrow[r, "{(x,y)}"'] & X \times Y \end{tikzcd}$$

With respect to this set-theoretic notation, we can formulate familiar properties of binary relations:

- A binary relation R on X is reflexive if $(x, x) \in_S R$ for any $x : S \to X$. Similarly we can formulate what does it mean for an internal binary relation on X to be symmetric, transitive, an equivalence, ect.
- An internal binary relation R from X to Y is difunctional if

$$\begin{bmatrix} u \\ y \end{bmatrix} \in_S R \quad \text{and} \quad \begin{bmatrix} u \\ v \end{bmatrix} \in_S R \quad \text{and} \quad \begin{bmatrix} x \\ v \end{bmatrix} \in_S R \implies \begin{bmatrix} x \\ y \end{bmatrix} \in_S R,$$

for any $x, u : S \to X$ and any $y, v : S \to Y$.

Remark

This general technique of working set-theoretically within abstract categories is formally justified by the Yoneda embedding $Y : \mathbb{C} \to \mathrm{Set}^{\mathbb{C}^{\mathrm{op}}}$

M-closedness

Definition

Given any matrix M with n rows, and any left exact category $\mathbb{C}$, we say that $\mathbb{C}$ has M-closed relations if for any n ary relation $r : R_0 \to X_1 \times \cdots \times X_n$ the relation on sets

$$\hom(S, R_0) \xrightarrow{\hom(S,r)} \hom(S, X_1) \times \cdots \times \hom(S, X_n)$$

is strictly M-closed for any object S in $\mathbb{C}$.

Mal'tsev categories

- A. Carboni, J. Lambek and M.C. Pedicchio, Diagram chasing in Mal'cev categories, Journal of Pure and Applied Algebra 69, 1990, 271–284.
- A. Carboni, M.C. Pedicchio and N. Pirovano, Internal graphs and internal groupoids in Mal'cev categories, Canadian Mathematical Society Conference Proceedings 1992, 97–109.

Definition

A *Mal'tsev category* may be simply defined as a category $\mathbb{C}$ in which every internal binary relation is difunctional.

Examples of Mal'tsev categories

- Any Mal'tsev (quasi)variety.
- The category Grp(Top) of topological groups.
- Internal Mal'tsev algebras in a finitely complete category.
- If $\mathbb{C}$ is a Mal'tsev category, then so is:
  - $\mathbb{C}^{\mathbb{D}}$ for any small category $\mathbb{D}$.
  - $\mathbb{C} \downarrow X$ and $X \downarrow \mathbb{C}$ for any object X in $\mathbb{C}$.

What about co-Mal'tsev categories?

We can also search for examples of categories $\mathbb{C}$ which satisfy the dual co-Mal'tsev property, i.e., such that $\mathbb{C}^{\mathrm{op}}$ is Mal'tsev.

Mal'tsev objects

An internal relation R in a category $\mathbb{C}$, represented by a monomorphism $r : R_0 \to X \times Y$ is called difunctional if and only if the relation given by the image of the function

$$\hom(S, R_0) \xrightarrow{\hom(S,r)} \hom(S, X) \times \hom(S, Y)$$

is a difunctional relation of sets.

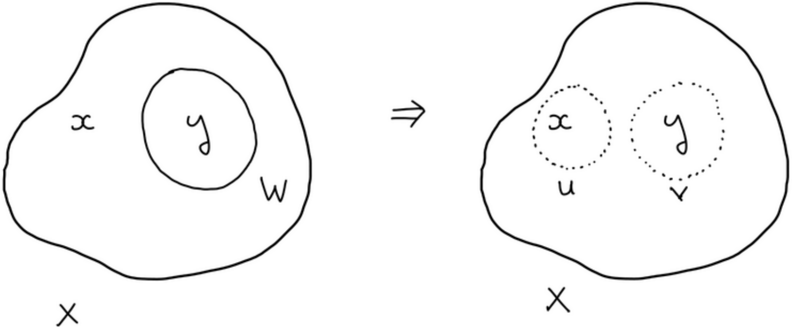
Definition (T. Weighill, 2017)

An object S is a category is called a Mal'tsev object if the relation (on sets) above is difunctional for any internal relation R as above.

A separation axiom

Definition

A topological space X is called preregular or $\mathbf{R}_1$ if for any two topologically distinguishable points are separable.



An exotic example of a Mal'tsev category

Theorem (T. Weighill, 2017)

A topological space S is a Mal'tsev object in $\mathbf{Top}^{\mathrm{op}}$ *if and only if it is an* $\mathbf{R_1}$*-space.*

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The category $\mathbf{R_1}$ *of* R_1*-spaces is such that* ${\mathbf{R_1}}^{\mathrm{op}}$ *is Mal'tsev. Moreover,* $\mathbf{R_1}$ *is the largest full subcategory of* $\mathbf{Top}$ *which is co-Mal'tsev under some additional requirements.*

The category of metric spaces

A map $f : X \to Y$ between metric spaces is called short if $d_Y(f(x), f(y)) \leqslant d_X(x, y)$, the category of metric spaces and short maps is denoted by **Met**.

Definition

A metric space (X, d) is an ultrametric space if it satisfies

$$d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$$

for any $x, y, z \in X$.

Example

The metric arising from the p-adic norm is an ultrametric.

Theorem (T. Weighill, 2017)

A metric space S is a Mal'tsev object in $\mathbf{Met}^{\mathrm{op}}$ *if and only if it is an ultrametric space. Moreover, the category of ultrametric spaces* **UMet** *forms a co-Mal'tsev category, i.e.,* $\mathbf{UMet}^{\mathrm{op}}$ *is Mal'tsev.*

- T. Weighill, Mal'tsev objects, R1-spaces and ultrametric spaces, Theory and Applications of Categories 32, 2017, 1485–1500.

Majority categories

Although $\mathbf{R}_1$ and **UMet** are co-Mal'tsev categories, the categories **Top** and **Met** are not. However, they are co-*majority categories*.

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Definition

A category $\mathbb{C}$ is a majority category if it satisfies the matrix property given by

$$\left[\begin{array}{ccc|c} x & x & y & x \\ x & y & x & x \\ y & x & x & x \end{array}\right]$$

- Hoefnagel, M.A.: Majority categories. Theory Appl. Categ. 34(10), 249–268 (2019)

Examples

Example

A variety is a majority category if and only if it admits a majority term, i.e., a ternary term $m(x, y, z)$ satisfying:

$$m(x, x, y) = x,$$
$$m(x, y, x) = x,$$
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\end{aligned}
$$

Example

Duals of categories of geometric structures tend to be majority categories. For example the duals of **Top**, **Met**, **Pos**, **Grph**, any topos, are majority categories.

Varieties admitting a majority term

Theorem (K.A. Baker and A.F. Pixley, 1975)

The following are equivalent for a variety $\mathbb{V}$ of algebras.

(i) $\mathbb{V}$ *admits a majority term.*

(ii) *Any subalgebra S of a finite product $A = A_1 \times A_2 \times \cdots \times A_n$ of algebras is uniquely determined by its images under the canonical projections $A \xrightarrow{\pi_{i,j}} A_i \times A_j$. This is to say that if S, T are any subalgebras of A, and if $\pi_{i,j}(S) = \pi_{i,j}(T)$ for all $i, j = 1, 2, ..., n$, then $S = T$.*

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- The notion of a majority category reformulates condition (i).
- If we are to reformulate (ii) then the base category should possess a suitable notion of image factorization of a morphism.

Regular epimorphisms

A morphism $q : X \to Q$ in a category $\mathbb{C}$ is called a *regular epimorphism* if it is a coequalizer of some parallel pair of morphisms in $\mathbb{C}$.

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A morphism $q : X \to Q$ in a category $\mathbb{C}$ is called a *regular epimorphism* if it is a coequalizer of some parallel pair of morphisms in $\mathbb{C}$. That is, there exists morphisms $A \to X$ making a coequalizer diagram:

$$A \rightrightarrows X \xrightarrow[q]{} Q$$

Example

Regular epimorphisms in a variety are precisely the surjective homorphisms in the variety.

Regular categories

Recall that a category $\mathbb{C}$ is said to be *regular* if following three properties:

(i) $\mathbb{C}$ is left exact, i.e., has all finite limits.

(ii) Every morphism in $\mathbb{C}$ factors as a regular-epimorphism followed by a monomorphism, i.e., every morphism admits an image factorisation.

(iii) The class of all regular epimorphisms in $\mathbb{C}$ is pullback stable, i.e., for any pullback diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "p"'] & \bullet \arrow[d, "e"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

if e is a regular epimorphism, then so is p.

The notion of a regular category provides us with a well behaved image factorisation, so that previous theorem generalises:

Theorem

The following are equivalent for a regular category $\mathbb{C}$.

(i) $\mathbb{C}$ *is a majority category.*

(ii) *Any subobject S of a finite product $A = A_1 \times A_2 \times \cdots \times A_n$ of algebras is uniquely determined by its images under the canonical projections $A \xrightarrow{\pi_{i,j}} A_i \times A_j$.*

The notion of a regular category provides us with a well behaved image factorisation, so that previous theorem generalises:

Theorem

The following are equivalent for a regular category $\mathbb{C}$.

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The category $\mathbf{Top}^{\mathrm{op}}$ is a regular majority category.

The Pairwise Chinese Remainder Theorem

An algebra A satisfies the *Pairwise Chinese Remainder Theorem* if for any congruences $\theta_1, \theta_2, ..., \theta_n$ on A and any elements $a_1, ..., a_n \in A$, if the system of congruences

$$x \equiv a_i \mod \theta_i \qquad \text{(for } i = 1, 2, ..., n\text{)}$$

is solvable two at a time, then it is solvable.

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is solvable two at a time, then it is solvable.

Theorem (K.A. Baker and A.F. Pixley, 1975)

For a variety of algebras $\mathbb{V}$, the following are equivalent:

- *$\mathbb{V}$ admits a majority term.*
- *Every algebra in $\mathbb{V}$ satisfies the Pairwise Chinese Remainder Theorem.*

The Pairwise Chinese Remainder Theorem (PCRT) can be formulated for arbitrary finitely complete categories, since it deals only with congruences. However:

First instance of context sensitivity

For a left exact category $\mathbb{C}$ we do not necessarily have that $\mathbb{C}$ is a majority category if and only if $\mathbb{C}$ satisfies the PCRT.

Arithmetical categories

There is a notion of *arithmetical category* introduced by M.C. Pedicchio:

- M.C. Pedicchio, Arithmetical categories and commutator theory, Applied Categorical Structures 4, 1996, 297–305.

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Definition

A Barr exact Mal'tsev category $\mathbb{C}$ (with coequalizers) is arithmetical if the lattice of equivalence relations on any object is distributive.

Theorem

A Barr exact Mal'tsev category $\mathbb{C}$ (with coequalizers) is arithmetical if and only if every internal groupoid in $\mathbb{C}$ is an equivalence relation.

Arithmetical categories

The matrix corresponding to a Pixley term

$$\left[\begin{array}{ccc|c} x & x & y & y \\ x & y & x & x \\ y & x & x & y \end{array}\right]$$

defines a matrix property such that if $\mathbb{C}$ is a Barr exact category with coequalizers, then $\mathbb{C}$ is arithmetical if and only if it satisfies the property corresponding to the matrix above.

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Example

A non-varietal example of an arithmetical category is given by the dual of any elementary topos.

Deciding implications of matrix properties

Matrix classes

Consider the matrix M where:

$$M = \left[\begin{array}{ccc|c} x_{11} & \dots & x_{1m} & y_1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \dots & x_{nm} & y_n \end{array}\right].$$

- mcvar$\{M\}$ = the class of all finitary varieties which have M-closed relations.
- mclex$\{M\}$ = the class of all left exact categories which have M-closed relations.

Matrix classes

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- $\mathrm{mcvar}\{M\}$ = the class of all finitary varieties which have M-closed relations.
- $\mathrm{mclex}\{M\}$ = the class of all left exact categories which have M-closed relations.

Question

Given two matrices M and N, can we have an algorithm for deciding whether $\mathrm{mclex}\{M\} \subseteq \mathrm{mclex}\{N\}$? And how does this problem compare to the corresponding algebraic problem of deciding whether $\mathrm{mcvar}\{M\} \subseteq \mathrm{mcvar}\{N\}$?

Example

Given any Mal'tsev term $p(x, y, z)$ which satisfies $p(x, y, x) = x$, i.e., any Pixley term, the term defined by

$$m(x, y, z) = p(x, p(x, y, z), z),$$

is a majority term.

Example

Given any Mal'tsev term $p(x,y,z)$ which satisfies $p(x,y,x) = x$, i.e., any Pixley term, the term defined by

$$m(x,y,z) = p(x, p(x,y,z), z),$$

is a majority term. Moreover, consider the matrices

$$\text{Pix} = \left[\begin{array}{ccc|c} x & x & y & y \\ x & y & x & x \\ y & x & x & y \end{array}\right], \quad \text{Maj} = \left[\begin{array}{ccc|c} x & x & y & x \\ x & y & x & x \\ y & x & x & x \end{array}\right]$$

then $\text{mclex}\{\text{Pix}\} \subseteq \text{mclex}\{\text{Maj}\}$.

Sharpness

Let M be any matrix, and suppose that $R \subseteq A_1 \times \cdots A_n$ is any n-ary relation.

Definition

R is said to be M-sharp if R is strictly M'-closed where M' is any matrix obtained by selecting any n rows of M.

Example

If $R = A_1 \times \cdots \times A_n$, then R is M-sharp.

Sharp-closure

We can always form the 'sharp closure' of the relation $R \subseteq A_1 \times \cdots A_n$ under the matrix M:

$$M(R) = \bigcap\{R' \mid R \subseteq R' \subseteq A_1 \times \cdots \times A_n \text{ and } R' \text{ is } M\text{-sharp}\}$$

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Recursively

In a similar way to how we would recursively generate subalgebras in a variety $M(R)$ may be presented as

$$M(R) = \bigcup_{i \in \mathbb{N}} R_i,$$

where $R_0 = R$ and R_{n+1} is obtained by adding to R_n the right columns of matrices M' whose left columns are in R_n and which are row-wise interpetations of the matrices formed from selecting n rows from M.

The algorithm

Theorem

Given any two (non-trivial) matrices N, M we have $\mathrm{mclex}\{N\} \subseteq \mathrm{mclex}\{M\}$ *if and only if the right column of M is contained in $N(R)$ where R is the set of left columns of M.*

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- The ‘if’ part of this statement is trivial because of the Yoneda embedding, the significant part of this statement is the ‘only if’ part.

Main ideas of the proof

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- Define the subcategory $M(\mathrm{Rel}_n)$ as those objects (X, R) for which R is M-sharp.
- The dual category $M(\mathrm{Rel}_n)^{\mathrm{op}}$ satisfies the matrix property M and is left-exact.
- In fact the category $M(\mathrm{Rel}_n)^{\mathrm{op}}$ satisfies many further exactness conditions, such as having colimits, image factorizations, ect.

Computer implementation

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We have implemented this algorithm on a computer in order to glimpse some fragments of the poset Mclex of all matrix classes. If $n, k > 0$ and $m \geqslant 0$ are integers then we will write $\mathrm{Mclex}[n, m, k]$ for the poset of all matrix classes determined by a matrix with at most n rows, m-columns and k distinct variables.

Representing arbitrary matrix properties by integer matrices

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Example

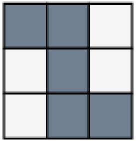
Consider the Pix matrix:

$$\left[\begin{array}{ccc|c} x & x & y & y \\ x & y & x & x \\ y & x & x & y \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

For visual purposes we represent matrices with cells with different colors. For example the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is represented by

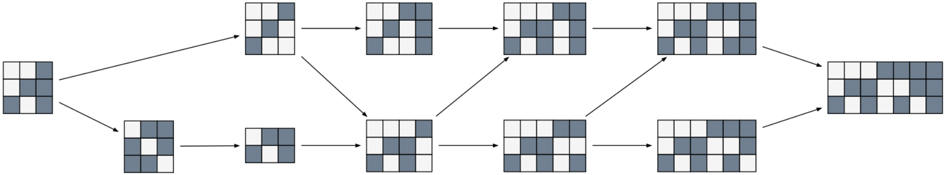


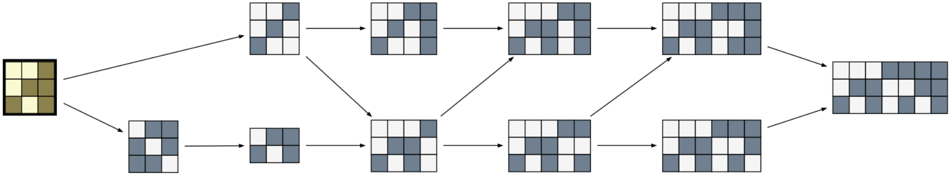
Double lexi-orderings

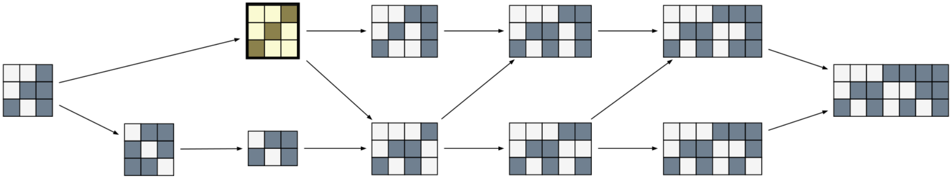
Rearranging columns and rows does not change the matrix property, so that we may also chose our matrices to have lexicographically ordered rows and columns.

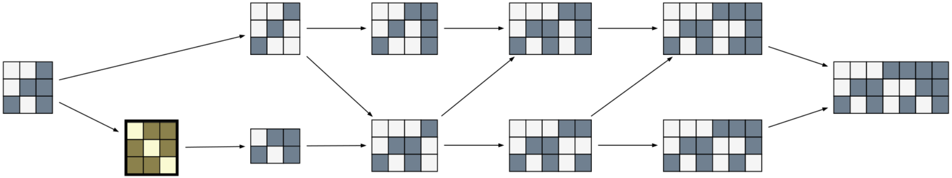
Mclex[3,7,2]

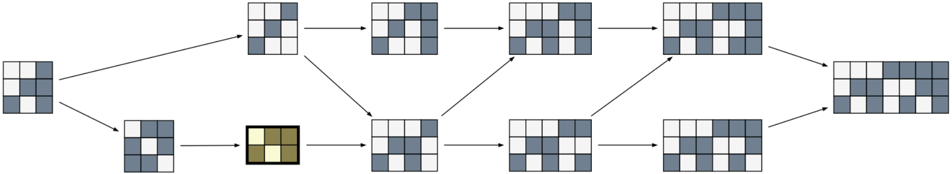
Matrices with at most 3 rows, 2 distict variables have (up to equivalence) at most $2^3 - 1$ columns.



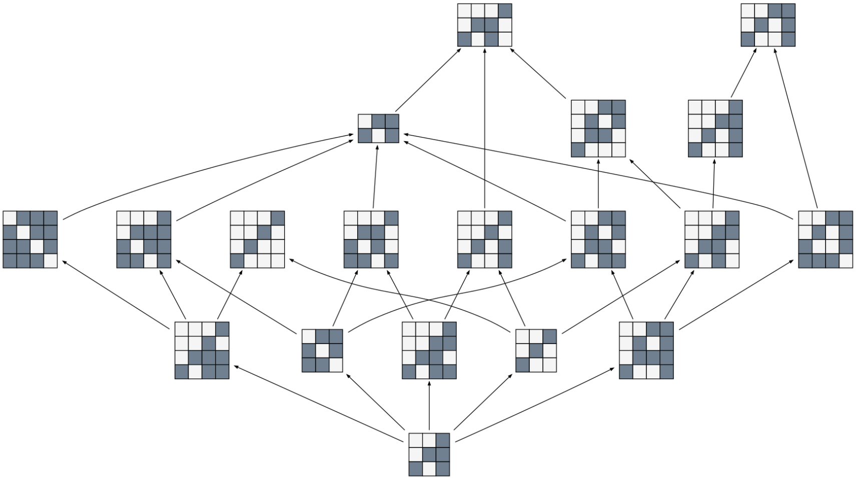








Mclex[4,4,2]



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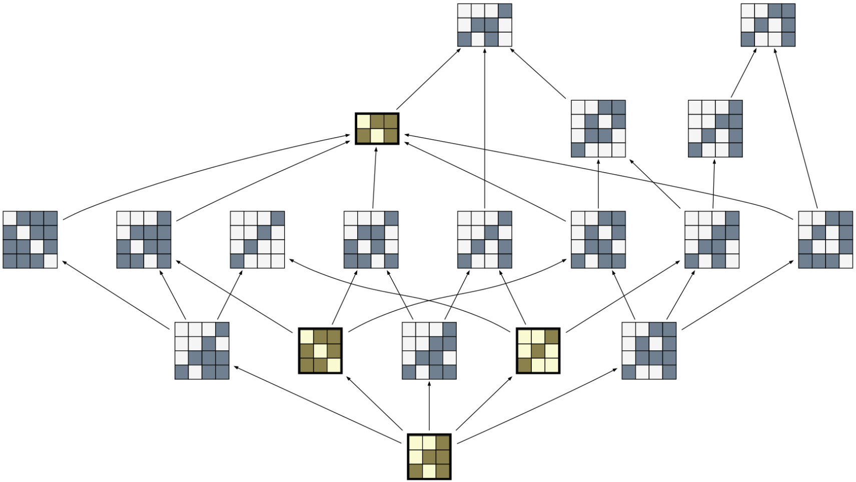

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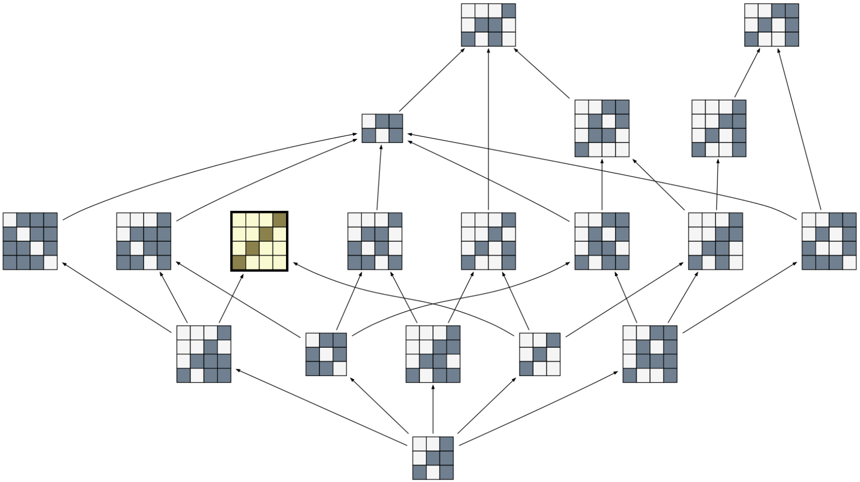
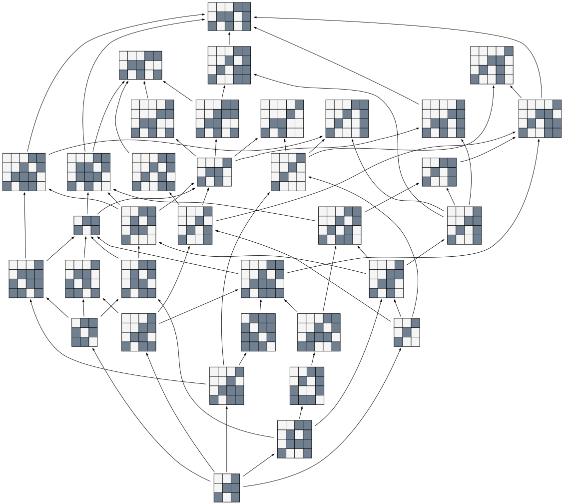
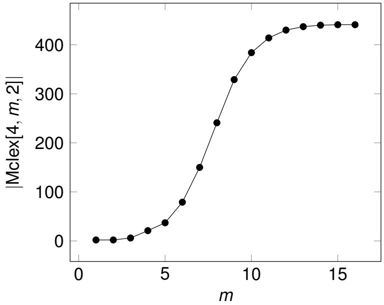
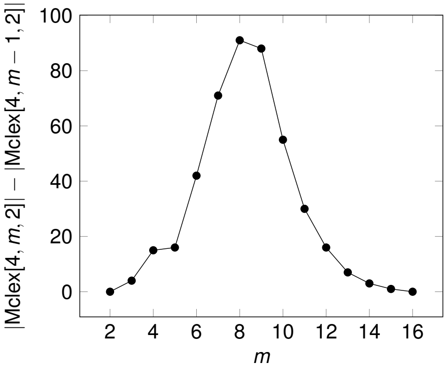


Mclex[4,5,2]



Count of matrix properties with at most four rows and two variables





$m =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15+
count	2	2	6	21	37	79	150	241	329	384	414	430	437	440	441

Near unanimity

A 4-ary near unanimity term is a 4-ary term p satisfying the equations:

$$
\begin{aligned}
p(x,x,x,y) &= x, \\
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Any variety which admits a near unanimity term is congruence distributive (A. Mitschke), and any congruence distributive Mal'tsev variety admits a majority term (A. Pixley).

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Corollary

A Mal'tsev variety which admits a 4-ary near unanimity term admits a majority term.

Context matters

This fact does not extend to left exact categories.

Context sensitivity

There are left exact Mal'tsev categories which satisfy the 4-near unanimity matrix property, but which are not majority categories.

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Remark

There are categories which are complete/cocomplete have pullback/pushout stable epimorphisms/monomorphisms, which are Mal'tsev and satisfy the 4-near unanimity matrix property, which are not majority categories.

Regular context

Remark

When the base category is regular, we do have that Mal'tsev + near unanimity gives majority.

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Another example of context sensitivity

Theorem

If $\mathbb{C}$ is any regular category which satisfies the arithmetical matrix property, then $\mathbb{C}$ satisfies any non-trivial matrix property.

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If $\mathbb{C}$ is any regular category which satisfies the arithmetical matrix property, then $\mathbb{C}$ satisfies any non-trivial matrix property.

- The result above applies in particular to the case when $\mathbb{C}$ is a variety.

Proof sketch

Regular majority categories have the following characterization:

Theorem (Equivalent to PCRT for regular categories)

Let $\mathbb{C}$ be any regular category, then the following are equivalent.

1. *For any non-trivial matrix M, if $\mathbb{C}$ satisfies the matrix property given by any selection of two rows from M, then $\mathbb{C}$ satisfies the matrix property M.*
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2. *$\mathbb{C}$ is a majority category.*

For (1) implies (2): every selection of two rows from the majority matrix

$$\left[\begin{array}{ccc|c} x & x & y & x \\ x & y & x & x \\ y & x & x & x \end{array}\right]$$

give a matrix property which every category satisfies.

Question

The above theorem motivates the question: how many non-trivial matrix properties are there with only two rows.

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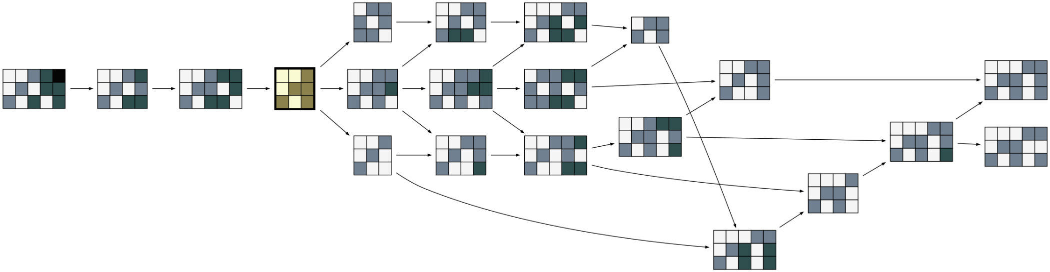
There is only one non-trivial matrix property with two rows, it is the Mal'tsev property.

- A category satisfies the Mal'tsev property and the majority property if and only if it satisfies the arithmetical matrix property.
- Any regular majority category which is Mal'tsev satisfies any non-trivial matrix property.

Mclex[3,5,4]



- In the left exact case there are matrices stronger than the arithmetical matrix.

Future questions

- Investigation of the results obtained in the lex context, rather in the context in of regular categories or varieties.
- Investigating matrix properties where the entries are a mixture of constants and variables. In particular, linking up the pointed case with what was presented here.
- Analagous results for other left-exact properties such as shifting conditions (shifting lemma, triangular lemma, trapezoid lemma, ect)

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