The Hidden Subgroup Problem for Universal Algebras

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1. Quantum Computation
2. Hidden Subgroups, Hidden Kernels
3. The Hidden Kernel Problem for Post's Lattice

Quantum Computation

Compared to classical computers, quantum computers are

- based on a different model of computation,
- very hard/maybe impossible to build at scale,
- very hard to program and reason about.

Classical computers are

- based on well-studied model of computation,
- cheap/easy to build,
- "easy" to program and reason about,
- fast (approx. exponential growth in speed).

Why bother?

Physical constraints: we will probably never have

- clock speeds faster than the electron transition frequency ($\approx 10^{15}$ Hz),
- components smaller than the diameter of a hydrogen atom ($\approx 10^{-8}$ cm).

Even in the distant future, classical computers will continue to struggle with problems of exponential complexity.

Idea: exploit natural phenomena to aid in computation.

- Quantum phenomena are hard to exploit...

How about using classical phenomena? Maybe use an analog co-processor?

- Classical phenomena can be simulated in polynomial time and space.
- Speedup will be at most polynomial.
- Classical physics is too "easy" to be useful.

Consider a quantum system of n particles with spins 0 or 1.

- Each particle is modelled by vector space $\mathfrak{B} = \mathbb{C}\text{-span}\{|0\rangle, |1\rangle\}$.
- Total system is modelled by 2^n-dimensional vector space,

$$\mathfrak{B}^{\otimes n} = \mathbb{C}\text{-span}\left\{ |s_1 \cdots s_n\rangle \mid s_i \in \{0,1\}\right\}.$$

- Possible states of the system are norm 1 vectors,

$$\sum_{s_1,\ldots,s_n \in \{0,1\}} \lambda_{s_1 \cdots s_n} |s_1 \cdots s_n\rangle = |\alpha\rangle .$$

- Probability of observing $|t_1 \cdots t_n\rangle$ when $|\alpha\rangle$ is measured $= |\lambda_{t_1 \cdots t_n}|^2$.
- Evolution over time is determined by action of $2^n \times 2^n$ unitary matrices.

Quantum systems represent exponentially difficult computational problems, in contrast to "easy" classical systems.

bit string $\longrightarrow$ Classical Computer $\longrightarrow$ bit string

- deterministic: equal inputs give equal outputs
- need not be reversible
- represented by a boolean circuit (for fixed length input)

bit string (as basis vector) $\longrightarrow$ Quantum Computer $\longrightarrow$ superposition of bit strings $= \sum \lambda_{s_1 \cdots s_n} |s_1 \cdots s_n\rangle$

measure $\longrightarrow$ bit string $s_1 \cdots s_n$ w/ probability $|\lambda_{s_1 \cdots s_n}|^2$

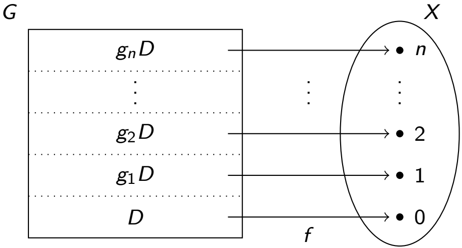
- probabilistic, likely need to run multiple times
- must be reversible (before measurement)
- represented by a quantum circuit (for fixed length input)

Hidden Subgroups, Hidden Kernels

Let $\mathbb{G}$ be a group, X a set, and $f : G \to X$ a function.

f hides a subgroup $\mathbb{D} \leq \mathbb{G}$ if f is constant precisely on $\mathbb{D}$-cosets.

$$f(a) = f(b) \iff aD = bD$$



The Hidden Subgroup Problem (HSP)

`Input:` $\mathbb{G}$, $f : G \to X$ hiding some subgroup **as a blackbox**

`Output:` the subgroup $\mathbb{D}$ that f hides (as generators)

Considerations

- Input size is $\lg(|G|)$.
- $\mathbb{D}$ must be specified with poly$(\lg(|G|))$ information.
- Clearly in $\mathcal{O}(|G|) = \mathcal{O}(2^{\lg(|G|)})$.
- Two kinds of complexity: circuit size, evaluations of f.

The Hidden Subgroup Problem (HSP)

`Input:` $\mathbb{G}$, $f : G \to X$ hiding some subgroup (as a blackbox):
$\left[f(a) = f(b) \Leftrightarrow aD = bD\right]$

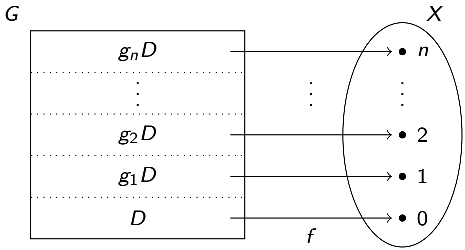
`Output:` the subgroup $\mathbb{D}$ that f hides (as generators)

Many famous problems are special cases of the HSP.

Problem	$\mathbb{G}$	Classical	Quantum
Simon's problem	$\mathbb{Z}_2^n$	$\Omega(2^n)$	$\mathcal{O}(n^2)$
Factoring	$\mathbb{Z}$	$\mathcal{O}(2^{K \lg(n)^{1/3} \lg\lg(n)^{2/3}})$	$\mathcal{O}(n^3)$
Discrete log	$\mathbb{Z} \times \mathbb{Z}$	$\mathcal{O}(2^{K \lg(n)^{1/3} \lg\lg(n)^{2/3}})$	$\mathcal{O}(n^3)$
Shortest vector	$\mathbb{D}_n$	$\mathcal{O}(2^{Kn})$	$\approx \mathcal{O}(8^{\sqrt{n}})$
Graph isomorphism	$\mathbb{S}_n$	$\mathcal{O}(2^{\lg(n)^K})$	

Polynomial-time quantum algorithms are known for

- abelian groups (Simon, Shor, Kitaev, et al),
- an irregular constellation of other groups.



Questions

- What makes abelian groups special?
- Can "hiding" a subgroup be made more natural?

Algebras

- An algebra is a set A together with operations $f_i : A^{n_i} \to A$ for $i \in I$, Written $\mathbb{A} = \langle A; \{f_i\}_{i \in I} \rangle$.
- Subalgebras don't form a meaningful partition of A, it's not clear how to define "hiding" a subalgebra.
- A congruence of $\mathbb{A}$ is a compatible equivalence relation. Equivalently, a congruence is the kernel of a homomorphism $\varphi : \mathbb{A} \to \mathbb{B}$.
- How does $f : A \to X$ hide a congruence θ of $\mathbb{A}$?

Hidden Kernel Problem (v1)

`Input`: $\mathbb{A}$, $f : A \to X$ hiding some congruence (as a blackbox)

`Output`: the congruence θ of $\mathbb{A}$ that f hides (as generators)

For $f : A \to X$, we say that f hides congruence θ of $\mathbb{A}$ if

$$f(a) = f(b) \qquad \iff \qquad a \, \theta \, b.$$

This allows us to impose algebraic structure on X: $\mathbb{A}/\theta$.

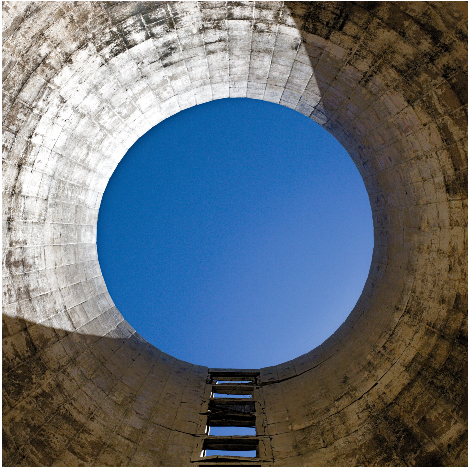
f is actually a homomorphism with kernel θ!

Hidden Kernel Problem

`Input`: algebras $\mathbb{A}$, $\mathbb{B}$, homomorphism $\varphi : \mathbb{A} \to \mathbb{B}$ (as a blackbox)

`Output`: generators of $\ker(\varphi)$

The Hidden Kernel Problem for Post's Lattice



Hidden Kernel Problem

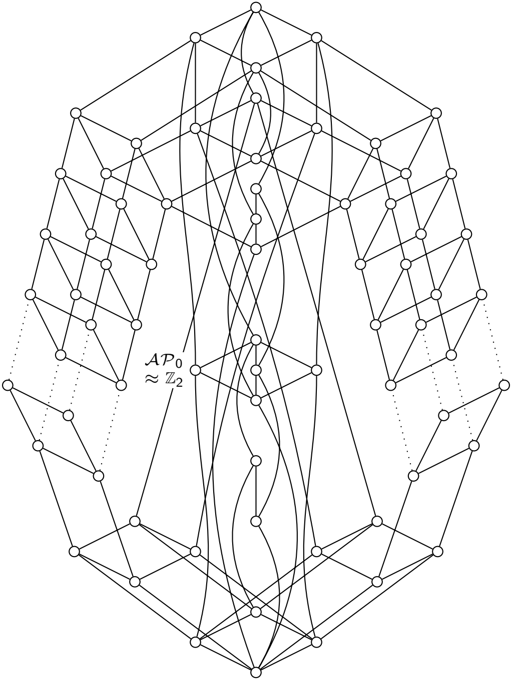
`Input`: homomorphism
$\varphi : \mathbb{A} \to \mathbb{B}$

`Output`: $\ker(\varphi)$ (generators)

Simon's algorithm solves this for $\mathbb{A} = (\mathbb{Z}_2)^n$.

How about if $\mathbb{A} = \mathbb{B}^n$, where $\mathbb{B}$ is a 2-element algebra?

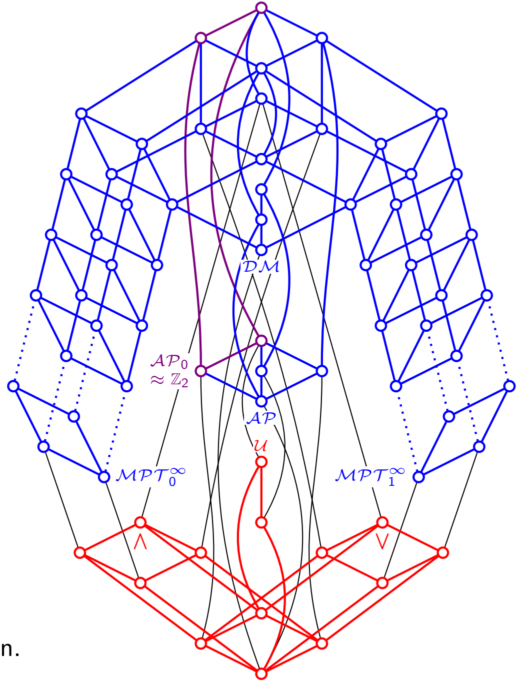
How many such $\mathbb{B}$ are there?



Theorem

There **exists** / **doesn't exist** *a poly-time* **quantum** *solution for HKP($\mathbb{B}^n$), where*

$\mathbb{B}$	*Ops on* $\{0,1\}$
$\mathcal{AP}_0$	$x+y$ (**known**)
$\mathcal{MPT}_0^\infty$	$x \wedge (y \vee z)$
$\mathcal{MPT}_1^\infty$	$x \vee (y \wedge z)$
$\mathcal{AP}$	$x+y+z$
$\mathcal{DM}$	$\mathrm{maj}(x,y,z)$
$\bigwedge$	$x \wedge y$, 0, 1
$\bigvee$	$x \vee y$, 0, 1
$\mathcal{U}$	$\neg x$, 0



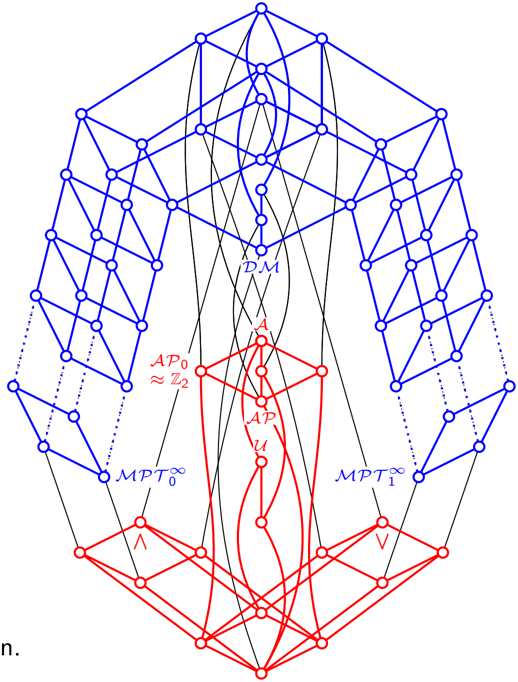
Observations

- **exists** is inherited up,
- **doesn't exist** is inherited down.

Theorem

There **exists** / **doesn't exist** *a poly-time* **classical** *solution for HKP*($\mathbb{B}^n$)*, where*

$\mathbb{B}$	*Ops on* $\{0,1\}$
$\mathcal{MPT}_0^\infty$	$x \wedge (y \vee z)$
$\mathcal{MPT}_1^\infty$	$x \vee (y \wedge z)$
$\mathcal{DM}$	$\mathrm{maj}(x,y,z)$
$\mathcal{A}$	$x \leftrightarrow y$, 0
$\bigwedge$	$x \wedge y$, 0, 1
$\bigvee$	$x \vee y$, 0, 1



Observations

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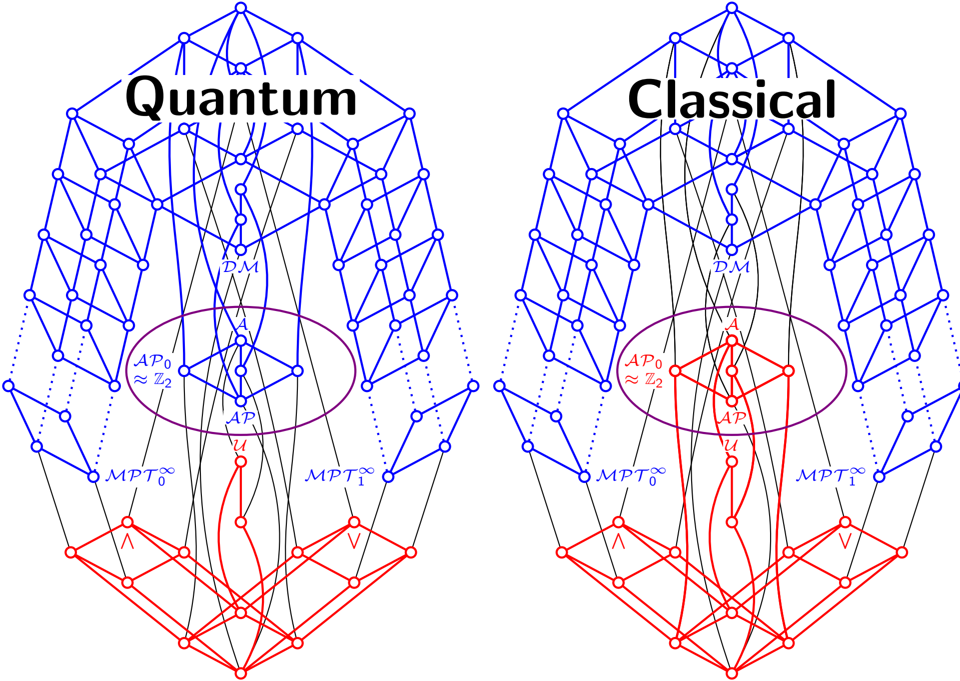
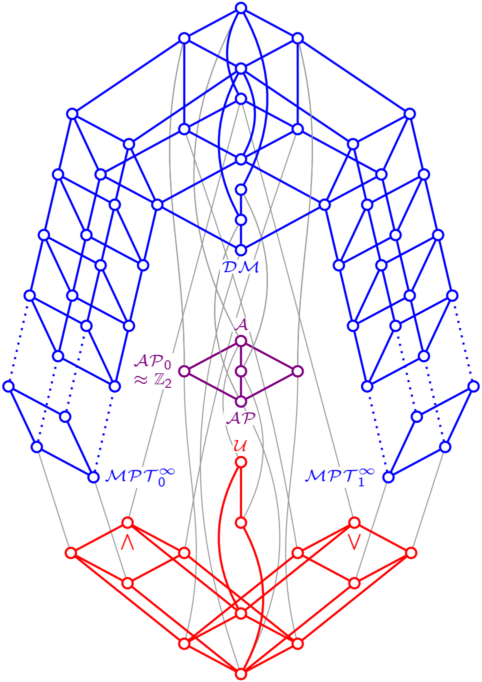

Quantum
Classical
$\mathcal{DM}$
$\mathcal{A}$
$\mathcal{AP}_0$
$\approx \mathbb{Z}_2$
$\mathcal{AP}$
$\mathcal{U}$
$\mathcal{MPT}_0^\infty$
$\mathcal{MPT}_1^\infty$
$\wedge$
$\vee$

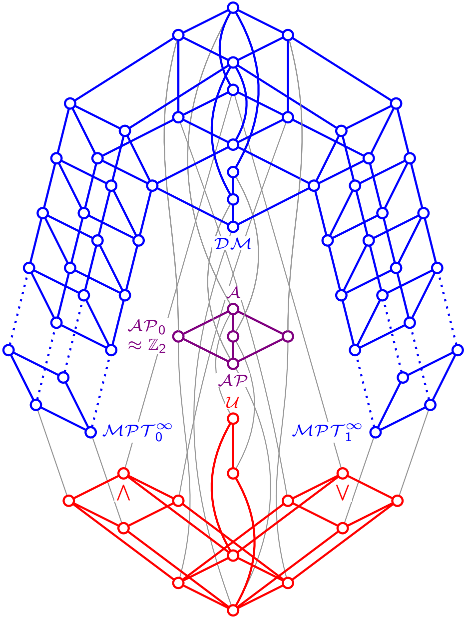


$\mathcal{DM}$
$\mathcal{A}$
$\mathcal{AP}_0$
$\approx \mathbb{Z}_2$
$\mathcal{AP}$
$\mathcal{U}$
$\mathcal{MPT}_0^\infty$
$\mathcal{MPT}_1^\infty$
$\wedge$
$\vee$

Theorem

Let $\mathbb{B}$ be a 2-element algebra and consider $HKP(\mathbb{B}^n)$.

- *If $\mathcal{MPT}_0^\infty$, $\mathcal{MPT}_1^\infty$, or $\mathcal{DM}$ is contained in $\mathbb{B}$ then classical and quantum poly-time solutions exist.*
- *If $\mathcal{AP} \preceq \mathbb{B} \preceq \mathcal{A}$ then a quantum poly-time solution exists while no classical poly-time one does.*
- *If $\mathbb{B}$ is contained in $\bigwedge$, $\bigvee$, or $\mathcal{U}$, then no poly-time quantum or classical solutions exist.*



Thank you for your attention.