Stability in abstract elementary classes of modules

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Background

- Abstract elementary classes were introduced by Shelah in the 70's.
- Shelah's eventual categoricity conjecture.
- The abstract theory has developed rapidly.
- Abelian groups and modules
- Stability.

Outline

1. **Basic notions**
2. Stability
3. Universal models

Basic notions: Model theory of modules

pp-formulas and *pp*-types

Let R be a ring and $L_R = \{0, +, -\} \cup \{r\cdot : r \in R\}$ be the language of R-modules.

- ϕ is a positive primitive formula if it is an existentially quantified finite system of linear equations.
- $pp(\bar{b}/A, N)$ are the *pp*-formulas satisfied by $\bar{b}$ in N with parameters in A.

Pure submodules

$M \leq_p N$ if and only if $M \subseteq N$ and $pp(\bar{a}/\emptyset, M) = pp(\bar{a}/\emptyset, N)$ for every $\bar{a} \in M^{<\omega}$.

For abelian groups:

$G \leq_p H$ if and only if $nH \cap G = nG$ for every $n \in \mathbb{N}$.

Basic notions: AECs

An *abstract elementary class* is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where K is a class of $\tau(\mathbf{K})$-structures and $\leq_{\mathbf{K}}$ is a partial order on K.

Key axioms

1. If $M \leq_{\mathbf{K}} N$, then M is a substructure of N.
2. Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:
   1. $M_\delta := \bigcup_{i<\delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.
   2. Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.
3. Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\mathrm{LS}(\mathbf{K})$ for the minimal such cardinal.

Examples

1. $(T, \preceq)$ for T a complete first-order theory.
2. $(Ab, \leq)$ and $(Ab, \leq_p)$.
3. $(\mathrm{TF}, \leq_p)$.
4. $(\mathrm{Tor}, \leq_p)$.
5. $(\aleph_1\text{-free}, \leq_p)$.
6. $(R\text{-Mod}, \subseteq_R)$ and $(R\text{-Mod}, \leq_p)$.
7. $(R\text{-Flat}, \leq_p)$.
8. $(R\text{-Absp}, \leq_p)$.
9. $(V, \subseteq)$ where V is a variety.
10. ...

Amalgamation property (AP)

Every $M \leq_{\mathbf{K}} N_1, N_2$ can be completed to a commutative square in $\mathbf{K}$.

$$\begin{tikzcd}
N_1 \arrow[r, "f"] & N' \\
M \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & N_2 \arrow[u, "g"']
\end{tikzcd}$$

Examples

- AP: $(R\text{-Mod}, \leq_p)$, $(R\text{-Absp}, \leq_p)$.
- No AP: $(\aleph_1\text{-free}, \leq_p)$.
- AP?: $(B_0, \leq_p)$.

Basic notions: AECs

Some properties

1. **K** has the *joint embedding property* (JEP): if every $M, N \in K$ can be **K**-embedded to a model in **K**.
2. **K** has *no maximal models* (NMM): if every $M \in K$ can be properly extended in **K**.

Examples

All the examples we introduced have JEP and NMM with the exception of Example 9.

Outline

Stability: Galois-types

We assume amalgamation for simplicity.

Pre-types

1. $\mathbf{K}^3 = \{(a, M, N) : M, N \in \mathbf{K}, M \leq_{\mathbf{K}} N \text{ and } a \in N\}$.
2. For $(a_1, M_1, N_1), (a_2, M_2, N_2) \in \mathbf{K}^3$, we say $(a_1, M_1, N_1)E(a_2, M_2, N_2)$ if:

$$\begin{tikzcd}
N_1 \arrow[r, "f_1"] & N \\
M = M_1 = M_2 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"] & N_2 \arrow[u, "f_2"']
\end{tikzcd}$$

$$f_1(a_1) = f_2(a_2)$$

Stability

Galois-types

1. For $(a, M, N) \in \mathbf{K}^3$, let $\mathbf{gtp}_{\mathbf{K}}(a/M; N) := [(a, M, N)]_E$.
2. $\mathbf{gS}(M)$ is the set of Galois-types over M.

T is a complete first-order theory: $(Mod(T), \preceq)$

$\mathbf{gtp}_{\mathbf{K}}(a/M; N) = \mathbf{gtp}_{\mathbf{K}}(b/M; N)$ if and only if $tp(a/M, N) = tp(b/M, N)$

Stable

- $\mathbf{K}$ is λ-stable if $|\mathbf{gS}(M)| \leq \lambda$ for all $M \in \mathbf{K}$ of cardinality λ.
- $\mathbf{K}$ is stable if there is a λ such that $\mathbf{K}$ is λ-stable.

Stability: Classes of modules

Theorem (Fisher-Bauer 70s)

If T is a complete first-order theory extending the theory of modules, then $(Mod(T), \leq_p)$ is stable.

Question 1

Let R be an associative ring with unity.
If $(K, \leq_p)$ is an AEC of modules, is $\mathbf{K}$ stable? Is this true if $R = \mathbb{Z}$? Under what conditions on R is this true?

Stability under Hypothesis 1

Hypothesis 1

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under direct summands.
3. K is closed under pure-injective envelopes, i.e., if $M \in K$, then $PE(M) \in K$.

Examples

- R-modules.
- Absolutely pure modules.
- Locally injective modules.
- Locally pure-injective modules.

Stability under Hypothesis 1

Lemma (M.)

Let $M, N_1, N_2 \in \mathbf{K}$, $M \leq_p N_1, N_2$, $b_1 \in N_1$ and $b_2 \in N_2$. Then:

$$\mathbf{gtp}(b_1/M; N_1) = \mathbf{gtp}(b_2/M; N_2) \text{ iff } \mathrm{pp}(b_1/M, N_1) = \mathrm{pp}(b_2/M, N_2).$$

Theorem (M.)

Let $\lambda \geq \mathrm{LS}(\mathbf{K})$. If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}$ is λ-stable.

Proof Sketch.

- Let $\{\mathbf{gtp}(a_i/M; N) : i < \alpha\}$ be an enumeration of $\mathbf{gS}(M)$.
- Let $\Phi : \mathbf{gS}(M) \to S_{pp}^{Th(N)}(M)$ be such that $\phi(\mathbf{gtp}(a_i/M; N)) = pp(a_i/M, N)$.
- Φ is an injective function and use first-order stability.

Stability under Hypothesis 2

Hypothesis 2

Let $\mathbf{K} = (K, \leq_p)$ be an AEC of modules such that:

1. K is closed under direct sums.
2. K is closed under pure submodules.
3. K is closed under pure epimorphic images: $ker(f) \leq_p dom(f)$.

Examples

1. R-modules.
2. Torsion-free groups.
3. Flat modules.
4. Abelian p -groups.
5. $\mathfrak{s}$-torsion modules.

Stability under Hypothesis 2

Galois-types=*pp*-types?

- I do not know.
- We can identify Galois-types and *pp*-types in the class of *p*-groups.

Theorem (M.)

If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}$ is λ-stable.

Stability under Hypothesis 2

Lemma

There is a non-forking relation on Galois-types that satisfies:

1. (Lieberman-Rosický-Vasey) (Uniqueness) If $M \leq_p N$, $p, q \in \mathbf{gS}(N)$, p, q do not fork over M and $p \restriction_M = q \restriction_M$, then $p = q$.
2. (M.) (Local character) If $p \in \mathbf{gS}(M)$, then there is $N \leq_p M$ such that p does not fork over N and $\|N\| \leq |R| + \aleph_0$.

Proof sketch.

- Let P be the pushout of $(i : M_0 \to M_1, j : M_0 \to M_2)$ in $R\text{-}Mod$.
- $M_1 \underset{M_0}{\overset{N'}{\downarrow}} M_2$ if the unique map $t : P \to N'$ is a pure embedding.
- One extends this to Galois-types.

Stability for classes of torsion-free abelian groups

Hypothesis 3

Let $\mathbf{K} = (K, \leq_p)$ be an AEC with $K \subseteq TF$ such that:

1. K has arbitrarily large models.
2. K is closed under pure submodules.

Examples

- Torsion-free abelian groups.
- $\aleph_1$-free abelian groups.
- Finitely Butler groups.

Stability for classes of torsion-free abelian groups

Lemma (M.)

If $\lambda^{|R|+\aleph_0} = \lambda$, then $\mathbf{K}$ is λ-stable.

Proof sketch. One identifies Galois-types in the class with Galois-types in the class of torsion-free abelian groups.

Why does it works for classes of torsion-free groups?

If $N_1, N_2 \leq_p N$, then $N_1 \cap N_2 \leq_p N$.

Theorem (M.)

Assume R is Von Neumann regular. If $\mathbf{K}$ is closed under submodules and has arbitrarily large models, then $\mathbf{K}$ is λ-stable for every λ such that $\lambda^{|R|+\aleph_0} = \lambda$.

Outline

1. Basic notions
2. Stability
3. **Universal models**

Universal models

Notation: $\mathbf{K}_\lambda = \{M \in K : M \text{ has cardinality } \lambda\}$

Universal model

- $M \in \mathbf{K}$ is a *universal model in* $\mathbf{K}_\lambda$ if $M \in \mathbf{K}_\lambda$ and if given any $N \in \mathbf{K}_\lambda$, there is a $\mathbf{K}$-embedding $f : N \to M$, i.e., $f : N \cong f[N] \leq_{\mathbf{K}} M$.
- We say that $\mathbf{K}$ has a universal model of cardinality λ if there is a universal model in $\mathbf{K}_\lambda$.

Example

$(\mathbb{Q}\text{-VS}, \subseteq_{\mathbb{Q}})$: For every λ, $\mathbb{Q}^{(\lambda)}$ is a universal model of size λ.

Universal models in (p-groups, $\leq_p$)

Abelian p-groups

G is a p-group if every element $g \neq 0$ has order p^n for some $n \in \mathbb{N}$.

Example

(p-groups, $\subseteq$): For every λ, $\mathbb{Z}(p^\infty)^{(\lambda)}$ is a universal model of size λ.

Question 2 (*Abelian groups* by L. Fuchs)

For which cardinals λ, does (p-groups, $\leq_p$) has a universal model of cardinality λ? The same question for torsion-free abelian groups with pure embeddings.

Answer under GCH

There is a universal model for every uncountable cardinal.

Universal models in (p-groups, $\leq_p$)

Theorem (M.)

If $\lambda^{\aleph_0} = \lambda$ or $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$, then ($p$-groups, $\leq_p$) has a universal model of cardinality λ.

Proof sketch.

- If $\lambda^{\aleph_0} = \lambda$, then ($p$-groups, $\leq_p$) is λ-stable.
- (p-groups, $\leq_p$) has AP, JEP and NMM.
- (Kucera-M.) (p-groups, $\leq_p$) has a universal model of cardinality λ.

Lemma (Kucera-M.)

Let **K** be an AEC with AP, JEP and NMM. Assume there is a κ such if $\theta^\kappa = \theta$, then **K** is θ-stable.
If $\lambda^\kappa = \lambda$ or $\forall \mu < \lambda(\mu^\kappa < \lambda)$, then **K** has a universal model of size λ.

Universal models in (p-groups, $\leq_p$)

Lemma (M.)

Let λ be a regular cardinal and μ be a regular cardinal. If $\mu^+ < \lambda < \mu^{\aleph_0}$, then ($p$-groups, $\leq_p$) does not have a universal model of cardinality λ.

An answer below $\aleph_\omega$ without $\aleph_0$ and $\aleph_1$

For $n \geq 2$, (p-groups, $\leq_p$) has a universal model of cardinality $\aleph_n$ if and only if $2^{\aleph_0} \leq \aleph_n$.

Proof sketch.

- $\Leftarrow$: $\aleph_n^{\aleph_0} = \aleph_n$.
- $\Rightarrow$: $\aleph_0^+ < \aleph_n < \aleph_n^{\aleph_0}$.

Remark

1. There are partial solutions for $\aleph_1$.
2. The problem is wide open for $\aleph_0$, $\aleph_\omega$, and singular cardinals.

References

- Marcos Mazari-Armida, *Some stable non-elementary classes of modules*, submitted, 20 pages. URL: https://arxiv.org/abs/2010.02918
- Marcos Mazari-Armida, *A model theoretic solution to a problem of László Fuchs*, Journal of Algebra **567** (2021), 196–209.

Thank you!